Algorithms and Probability **Session Sheet S3** **FS 2026**

For the exercise sessions on 05 March 2026.

Exercise S3.1 – *Bipartite Matching*

Let $G = (A \dot\cup B, E)$ be a bipartite graph.

(a) Prove or disprove: If G has a Hamiltonian cycle, then G contains two disjoint perfect matchings.

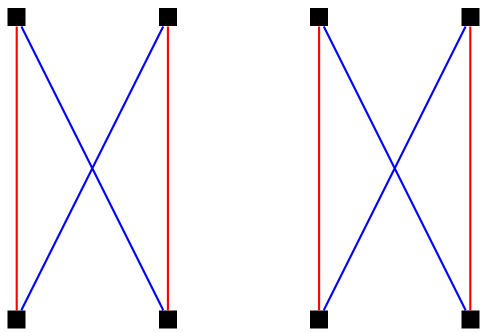
(b) Prove or disprove: If G contains two disjoint perfect matchings, then G has a Hamiltonian cycle.

(c) Let $A' \subseteq A$ and $B' \subseteq B$. Assume that there is a matching M_A covering A' and a matching M_B covering B' (these two matchings do not have to be disjoint!). Show that there is a matching M that covers both A' and B' (i.e. it covers $A' \cup B'$). (*Hint:* Which properties has the graph $(V, M_A \cup M_B)$? Try to build a matching using only edges in $M_A \cup M_B$.)

Solution S3.1 – *Bipartite Matching*

(a) The statement is true. Let C be a Hamiltonian cycle of G. Since G is bipartite every cycle (in particular C) has even length. Thus, C can be partitioned into two matchings by choosing every second edge for one matching and the remaining edges for the other matching.

(b) The statement is false, as the following examples shows. In red and blue two disjoint perfect matchings are shown. However, the graph cannot contain a Hamiltonian cycle as it is not connected.



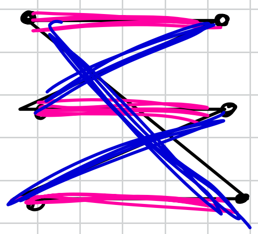
(c) As seen in the lecture, $M_A \cup M_B$ consists of vertex disjoint cycles and paths. We define a matching M as follows. For each cycle we choose every second edge for our matching M (as in (a)). This ensures that all vertices of cycles are covered by M. For paths of odd length (odd number of edges), we choose every second edge, starting with the first. This again ensures that all edges of such paths are covered by M (because we choose both the first and the last edge of the path). It remains to consider paths of even length. Let P be such a path. We need two properties of P. First, the edges of P are alternatingly part of M_A and M_B. Since there is an even number of edges, we may assume without loss of generality that the first edge is in M_A and the last edge is in M_B. Second, the vertices of P are alternatingly in A and in B. Since there is an odd number of vertices, we may assume without loss of generality that both the start and endpoint of P are in A. Hence, the endpoint of P is in A but it is only covered by an edge in M_B. Thus, it cannot be part of $A' \cup B'$, meaning that we do not have to cover it. Hence, choosing every second edge of P starting with the first (i.e. choosing $P \cap M_A$) covers all *relevant* vertices of P.

Exercise S3.1 – *Bipartite Matching*

Let $G = (A \dot\cup B, E)$ be a bipartite graph.

(a) Prove or disprove: If G has a Hamiltonian cycle, then G contains two disjoint perfect matchings.

Cycle is even !



(b) Prove or disprove: If G contains two disjoint perfect matchings, then G has a Hamiltonian cycle.