Distribution Examples

1) A student is taking a multiple-choice quiz with 10 questions. Each question has 4 options, only one is correct. The student guesses all answers randomly.

Wichtige Verteilungen

Name	Bezeichnung	Wertebereich	Dichte	Erwartungswert	Varianz
Bernoulli	Bernoulli(p)	$\{0,1\}$	$f_X(i) = \begin{cases} p & \text{für } i=1, \\ 1-p & \text{für } i=0. \end{cases}$	p	$p(1-p)$
Binomial	Bin(n,p)	$\{0,1,\ldots,n\}$	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometrisch	Geo(p)	$\mathbb{N}$	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	$\mathbb{N}_0$	$f_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	λ	λ

$$X \sim \mathrm{Bin}(10, 1/4)$$

1.1) What is the probability that the student gets exactly 3 correct?

$$\Pr[X=3] = \binom{10}{3} \cdot \left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^{10-3} \approx 0.250$$

1.2) What is the probability that the student gets at most 2 correct?

$$\Pr[X \leq 2] = \Pr[0] + \Pr[1] + \Pr[2] \approx 0.526$$

1.3) What is the expected number of correct answers?

$$\mathbb{E}[X] = n \cdot p = 10 \cdot 0.25 = 2.5$$

2) A person flips a fair coin repeatedly until they get heads for the first time.

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$$X \sim \mathrm{Geo}\left(\frac{1}{2}\right)$$

2.1) What is the probability that the first head occurs on the 3rd flip?

$$\Pr[X=3] = \frac{1}{2} \cdot \left(1-\frac{1}{2}\right)^{3-1} = 0.125$$

2.2) What is the probability that it takes more than 4 flips to get the first head?

$$\Pr[X>4] = \left(1-\frac{1}{2}\right)^4 = 0.0625$$

2.3) What is the expected number of flips to get the first head?

$$\mathbb{E}[X] = \frac{1}{p} = \frac{1}{1/2} = 2$$

3) A basketball player makes a free throw with a probability of 0.6. She continues shooting until she makes 3 successful shots.

2 successes

S

Bin(4,2) · p

$X \sim \text{NegativeBinomial}(3)$

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Poisson	Po(λ)	$\mathbb{N}_0$	$f_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	λ	λ

$\text{Bin}(4, 2)$

3.1) What is the probability that she takes exactly 5 shots to make her 3rd success?

$\binom{4}{2} \cdot (0.6)^2 \cdot (0.4)^2 \cdot p$

$$\Pr[X = 5] = \binom{4}{2} \cdot (0.4)^2 \cdot (0.6)^3 \approx 0.207$$

3.2) What is the probability that she needs more than 6 shots to get 3 successful shots?

$$\Pr[X > 6] = 1 - \Pr[X \leq 5]$$

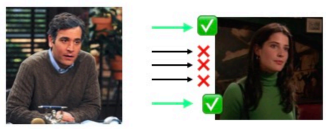
$$= 1 - (\Pr[X=3] + \Pr[X=4] + \Pr[X=5])$$

3.3) What is the expected number of shots to make 3 successes?

$$E[X] = \frac{n}{p} = \frac{3}{0.6} = 5$$

Probability Theory

Negative Binomial Distribution



#yes-no questions needed to get n yesses

$$X \sim \text{NegativeBinomial}(n)$$

$$f_X(k) = \begin{cases} \binom{k-1}{n-1}(1-p)^{k-n}p^n, & \text{for } k = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

Example to remember:

Coin toss until n-th head comes

X = #tosses

$$E[X] = n/p$$

4) A cereal company includes one of 5 different Pokémon stickers randomly in each cereal box. Each sticker is equally likely. You want to collect all 5.

Coupon Collector

$$X_i \sim Geo\left(\frac{5-(i-1)}{5}\right)$$

4.1) What is the expected number of cereal boxes you need to buy to collect all 5 different stickers?

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot H_n = 5 \cdot \underbrace{\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)}_{=H_5}$$

$$5 \cdot H_5$$

4.2) What is the expected number of boxes to get the last missing sticker?

$$\mathbb{E}[X_5] = \frac{1}{p} = 5$$