- Elementary Event ω_i (Elementarereignis) : A single outcome of the experiment
- Sample Space Ω (Ergebnismenge) : The set of all possible outcomes $\Omega = \{\omega_1, \omega_2, \dots\}$
- Elementary Probability $Pr[\omega_i]$ (Elementarwahrcheinlichkeit) : Probability of an elementary event
- Event E (Ereignis) : A subset of the sample space ($E \subseteq \Omega$)
- Complementary event $\overline{E}$ (Komplementärereignis) : All outcomes not in E $\overline{E} := \Omega \setminus E$

- $0 \leq Pr[\omega_i] \leq 1$
- $\sum_{\omega \in \Omega} \Pr[\omega] = 1$
- Probability of an event E : $Pr[E] := \sum_{\omega \in E} \Pr[\omega]$

- For all events $A, B, A_1, A_2, \dots$
  - $Pr[\emptyset] = 0, Pr[\Omega] = 1$
  - $0 \leq Pr[A] \leq 1$
  - $Pr[\overline{A}] = 1 - Pr[A]$
  - $A \subseteq B \implies Pr[A] \leq Pr[B]$

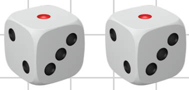
w_i ? 4

Ω ? $= \{1, 2, 3, 4, 5, 6\}$

$Pr[w_i]$? $1/6$

A := "Outcome is even" $Pr[A]$?

$3/6$



w_i ? $(1, 2)$

Ω ? $\Omega := \{(x_1, x_2) \mid x_1, x_2 \in \{1,2,3,4,5,6\}\}$

$Pr[w_i]$? $1/36$

A: "The sum of the outcomes is 4" $Pr[A]$?

$(1,3)$
$(3,1)$
$(2,2)$

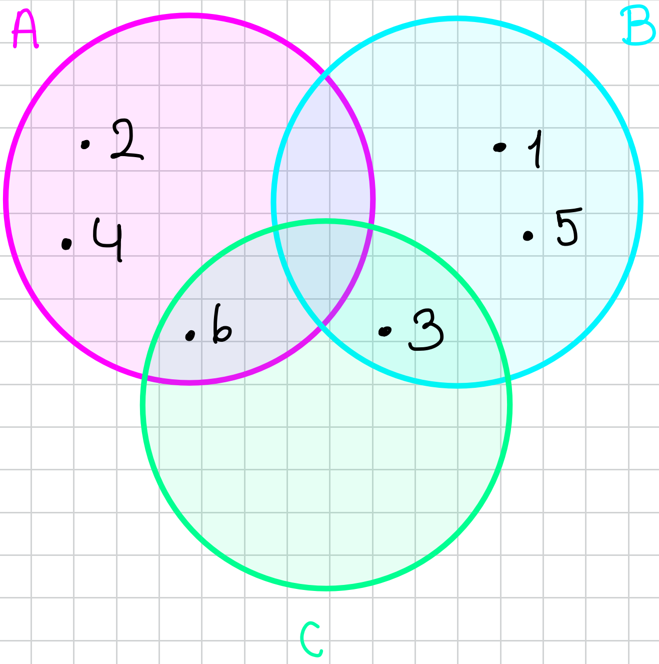
$3/36$

$$\Omega = \{1,2,3,4,5,6\}$$

A: " Outcome is even "

B: " Outcome is odd "

C: " Outcome % 3 = 0 "



Addition Rule

Events $A_1, \dots, A_n$ are pairwise disjoint

$$\implies \Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i]$$

Boolean Inequality

For arbitrary events $A_1, \dots, A_n$

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \Pr[A_i]$$

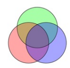
pairwise disjoint : For all pairs A_i, A_j with $i \neq j$: $A_i \cap A_j = \emptyset$

Inclusion-Exclusion Principle (Siebformel)

For events $A_1, \dots, A_n$

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{1 \leq i_1 < i_2 \leq n} \Pr[A_{i_1} \cap A_{i_2}] + \cdots$$
$$+ (-1)^{l+1} \sum_{1 \leq i_1 < \cdots < i_l \leq n} \Pr[A_{i_1} \cap \cdots \cap A_{i_l}] + \cdots$$
$$+ (-1)^{n+1} \Pr[A_1 \cap \cdots \cap A_n] .$$

n=3:



$\Pr[A \cup B \cup C]$
$= \Pr[A] + \Pr[B] + \Pr[C]$
$- \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C] + \Pr[A \cap B \cap C]$

7

$$\Pr[\text{"A or B"}] = \Pr[A \cup B] = \Pr[A] + \Pr[B] = \underline{1}$$

$$\Pr[\text{"B or C"}] = \Pr[B \cup C] = \Pr[B] + \Pr[C] - \Pr[B \cap C]$$
$$= 3/6 + 2/6 - 1/6 = 4/6$$

$$\Pr[\text{"A or B or C"}] = \Pr[A] + \Pr[B] + \Pr[C]$$
$$- \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C]$$
$$+ \Pr[A \cap B \cap C]$$

$$= \underline{1}$$

Probability Theory
Conditional Probability

the probability that event A will occur if we already know that event B has occurred

- conditional probability :
  - Let A and B be arbitrary events with $Pr[B] > 0$. The conditional probability $Pr[A \mid B]$ of A given B is

$$Pr[A \mid B] := \frac{Pr[A \cap B]}{PR[B]}$$

$$Pr[A \cap B] = Pr[A \mid B] \cdot Pr[B]$$
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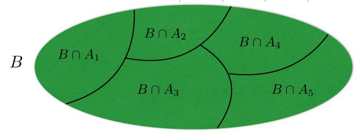


Bedingte Wahrscheinlichkeiten

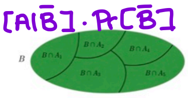
- **Definition:** Ist $\Pr[B] > 0$, so ist $\Pr[A|B] := \frac{\Pr[A \cap B]}{\Pr[B]}$.
- **Multiplikationssatz:** Ist $\Pr[A_1 \cap \ldots \cap A_n] > 0$, so ist

$$\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdot \ldots \cdot \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

$Pr[A \cap B] = Pr[A] \cdot Pr[B|A]$

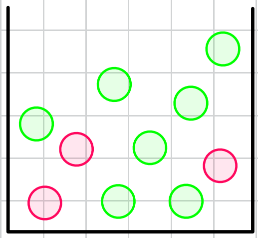
- **Satz von der totalen Wahrscheinlichkeit:** Ist $\Omega = A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n] > 0$, so gilt $\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i]$.

$Pr[A] = Pr[A|B] \cdot Pr[B] + Pr[A|\bar{B}] \cdot Pr[\bar{B}]$



- **Satz von Bayes:** Ist $B \subseteq A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n], \Pr[B] > 0$, so gilt

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$



7 green

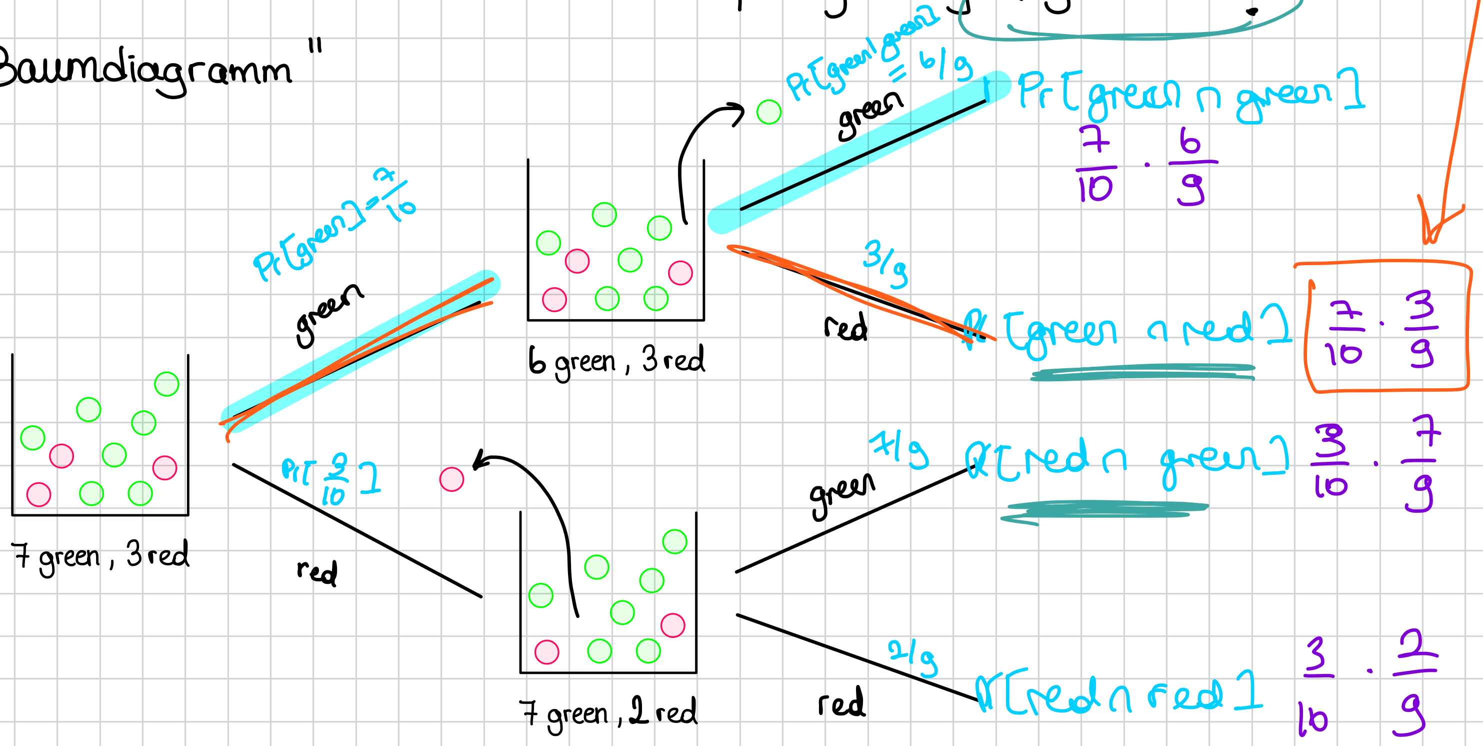
3 red

Pick 2 without putting back.

1) What is the probability of picking green and red afterwards ?

2) What is the probability of picking only 1 green ball ?

"Baumdiagramm"



Probability Theory
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the probability that event A will occur if we already know that event B has occurred

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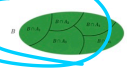

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$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$

$$Pr[A|B] = \frac{Pr[B|A] \cdot Pr[A]}{Pr[B]}$$

Let's look at a practical example. Imagine a fellow student of yours is stopped by the police after a night out and has to take a breathalyzer test. The test detects alcohol consumption in 99.9% of cases.

The test detects alcohol consumption in 99.9% of cases.

However, it also produces a positive result in 3% of cases, even though the person being tested hasn't consumed any alcohol.

We also know that 5% of the people tested have actually consumed alcohol:

The test is positive for your fellow student. What is the probability that they have actually consumed alcohol?

+ / −   alcohol / no alcohol

$$Pr[+ \mid alcohol] = 99.9/100$$
$$Pr[+ \mid no\ alcohol] = 3/100$$
$$Pr[alcohol] = 5/100$$

$$Pr[alcohol \mid +] = ?$$

Satz von Bayes:

$$Pr[A|B] = \frac{Pr[B|A] \cdot Pr[A]}{Pr[B]}$$

$$Pr[alcohol \mid +] = \frac{Pr[+ \mid alcohol] \cdot Pr[alcohol]}{Pr[+]}$$

$$Pr[+] = Pr[+ \mid alcohol] \cdot Pr[alcohol] + Pr[+ \mid no\ alcohol] \cdot Pr[no\ alcohol]$$