Wald's Identity

Probability Theory
Wald's Identity

- N and X are two independent random variables, $W_N \subseteq \mathbb{N}$

$$Z := \sum_{i=1}^{N} X_i \quad \text{where } X_1, X_2 \dots \text{ are independent copies of } X$$

$$\mathbb{E}[Z] = \mathbb{E}[N] \cdot \mathbb{E}[X]$$

Wichtige Verteilungen

Name	Bezeichnung	Wertebereich	Dichte	Erwartungswert	Varianz
Bernoulli	Bernoulli(p)	$\{0,1\}$	$f_X(i) = \begin{cases} p & \text{für } i=1, \\ 1-p & \text{für } i=0. \end{cases}$	p	$p(1-p)$
Binomial	Bin(n,p)	$\{0,1,\dots,n\}$	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometrisch	Geo(p)	$\mathbb{N}$	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	$\mathbb{N}_0$	$f_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	λ	λ

You're playing a game where each round:

You succeed with probability p and fail with probability 1 – p

- If you fail, you earn 0 CHF.
- If you succeed, you earn 10 CHF, and the game ends.

What's the expected number of total reward ?

$X_i \sim \text{Bernoulli}(p)$: (1 if success , 0 otherwise)

$R_i = 10 \cdot X_i$: (the reward from round i)

$N \sim \text{Geometric}(p)$: (number of trials until the first success)

$$S_N = \sum_{i=1}^{N} R_i$$

$$\mathbb{E}[S_N] = \underbrace{\mathbb{E}[N]}_{\frac{1}{p}} \cdot \mathbb{E}[R]$$

$\mathbb{E}[R_i] = 10 \cdot \mathbb{E}[X_i]$
$= 10 \cdot p$

$$= \frac{1}{p} \cdot 10p = 10$$