

Distribution Examples

- 1) A student is taking a multiple-choice quiz with 10 questions. Each question has 4 options, only one is correct. The student guesses all answers randomly.

$$X \sim \text{Bin}(10, 1/4)$$

Wichtige Verteilungen

Name	Bezeichnung	Wertebereich	Dichte	Erwartungswert	Varianz
Bernoulli	Bernoulli(p)	{0, 1}	$f_X(i) = \begin{cases} p & \text{für } i = 1, \\ 1-p & \text{für } i = 0. \end{cases}$	p	$p(1-p)$
Binomial	Bin(n, p)	{0, 1, ..., n}	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometrisch	Geo(p)	\mathbb{N}	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	\mathbb{N}_0	$f_X(i) = \frac{e^{-\lambda} \lambda^i}{i!}$	λ	λ

- 1.1) What is the probability that the student gets exactly 3 correct?

$$\Pr[X=3] = \binom{10}{3} \cdot \left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^{10-3} = 0.250$$

- 1.2) What is the probability that the student gets at most 2 correct?

$$\Pr[X \leq 2] = \Pr[0] + \Pr[1] + \Pr[2] \approx 0.526$$

- 1.3) What is the expected number of correct answers?

$$\mathbb{E}[X] = n \cdot p = 10 \cdot 0.25 = 2.5$$

- 2) A person flips a fair coin repeatedly until they get heads for the first time.

$$X \sim \text{Geo}\left(\frac{1}{2}\right)$$

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- 2.1) What is the probability that the first head occurs on the 3rd flip?

$$\Pr[X=3] = \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right)^{3-1} = 0.125$$

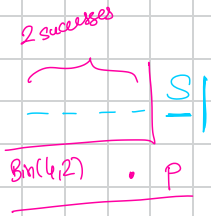
- 2.2) What is the probability that it takes more than 4 flips to get the first head?

$$\Pr[X > 4] = \left(1 - \frac{1}{2}\right)^4 = 0.0625$$

- 2.3) What is the expected number of flips to get the first head?

$$\mathbb{E}[X] = \frac{1}{p} = \frac{1}{1/2} = 2$$

3) A basketball player makes a free throw with a probability of 0.6. She continues shooting until she makes 3 successful shots.



$$X \sim \text{NegativeBinomial}(3)$$

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Binomial	$\text{Bin}(n, p)$	{0, 1, ..., n}	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometrisch	Geop(p)	\mathbb{N}	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	\mathbb{N}_0	$f_X(i) = \frac{\lambda^i}{i!} e^{-\lambda}$	λ	λ

$$\text{Bin}(4, 2)$$

3.1) What is the probability that she takes exactly 5 shots to make her 3rd success?

$$\binom{4}{2} \cdot (0.6)^2 \cdot (0.4)^2 \cdot p$$

$$\Pr[X=5] = \binom{4}{2} \cdot (0.4)^2 \cdot (0.6)^3 \approx 0.207$$

3.2) What is the probability that she needs more than 6 shots to get 3 successful shots?

$$\begin{aligned} \Pr[X > 6] &= 1 - \Pr[X \leq 5] \\ &= 1 - (\Pr[X=3] + \Pr[X=4] + \Pr[X=5]) \end{aligned}$$

3.3) What is the expected number of shots to make 3 successes?

$$E[X] = \frac{n}{p} = \frac{3}{0.6} = 5$$

Probability Theory

Negative Binomial Distribution

$$X \sim \text{NegativeBinomial}(n)$$

$$f_X(k) = \begin{cases} \binom{k-1}{n-1} (1-p)^{k-n} p^n, & \text{für } k = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = n/p$$



#yes-no questions
needed to get n yesses

Example to remember:
Coin toss until n-th head comes
 $X = \text{\#tosses}$

- 4) A cereal company includes one of 5 different Pokémon stickers randomly in each cereal box. Each sticker is equally likely. You want to collect all 5.

Coupon Collector

$$X_i \sim \text{Geo}\left(\frac{5 - (i-1)}{5}\right)$$

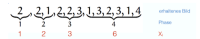
4.1) What is the expected number of cereal boxes you need to buy to collect all 5 different stickers?

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = n \cdot H_n = 5 \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = 5 \cdot H_5$$

4.2) What is the expected number of boxes to get the last missing sticker?

$$\mathbb{E}[X_5] = \frac{1}{p} = 5$$

Probability Theory Coupon Collector



phase i := turns while we have $i-1$ different coupons

X_i := #turns in phase i

$$X_i \sim \text{Geo}\left(\frac{n - (i-1)}{n}\right) \quad \mathbb{E}[X_i] = 1/p$$

$$X = \sum_{i=1}^n X_i$$

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n \frac{n}{n-i+1} = n \cdot \sum_{i=1}^n \frac{1}{i} = n \cdot H_n$$

collect all coupons and win

Example to remember:
n different coupons, we're getting one in each turn
X = #turns until we get all n coupons