ω_i ? 1

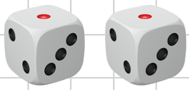
Ω ? $= \{1,2,3,4,5,6\}$

$Pr[\omega_i]$? $= 1/6$

$A :=$ "Outcome is even" $Pr[A]$?

2 4 6

$\frac{1}{2}$



ω_i ? $(1,2)$

Ω ? $= \{(x_1, x_2) \mid x_1, x_2 \in \{1,2,3,4,5,6\}\}$

$Pr[\omega_i]$? $1/36$

A: "The sum of the outcomes is 4" $Pr[A]$?

(1,3)

(2,2)

(3,1)

$3/36$

- Elementary Event ω_i (Elementarereignis) : A single outcome of the experiment
- Sample Space Ω (Ergebnismenge) : The set of all possible outcomes $\Omega = \{\omega_1, \omega_2, \dots\}$
- Elementary Probability $Pr[\omega_i]$ (Elementarwahrscheinlichkeit) : Probability of an elementary event
- Event E (Ereignis) : A subset of the sample space ($E \subseteq \Omega$)
- Complementary event $\overline{E}$ (Komplementärereignis) : All outcomes not in E $\overline{E} := \Omega \setminus E$

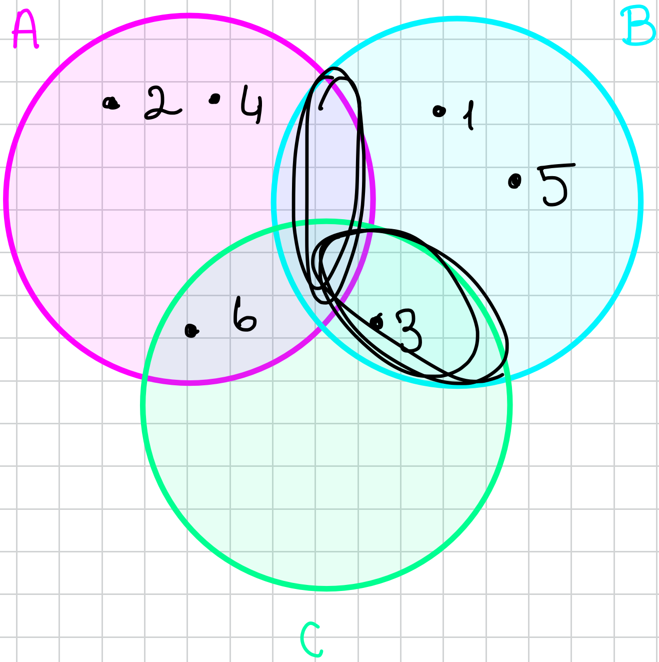
- $0 \leq Pr[\omega_i] \leq 1$
- $\sum_{\omega \in \Omega} \Pr[\omega] = 1$
- Probability of an event E : $Pr[E] := \sum_{\omega \in E} \Pr[\omega]$

- For all events $A, B, A_1, A_2, \dots$
  - $Pr[\emptyset] = 0$, $Pr[\Omega] = 1$
  - $0 \leq Pr[A] \leq 1$
  - $Pr[\overline{A}] = 1 - Pr[A]$
  - $A \subseteq B \implies Pr[A] \leq Pr[B]$

A : " Outcome is even "

B : " Outcome is odd "

C : " Outcome % 3 = 0 "



Addition Rule

Events $A_1, \dots, A_n$ are pairwise disjoint

$$\implies \Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i]$$

Boolean Inequality

For arbitrary events $A_1, \dots, A_n$

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \Pr[A_i]$$

pairwise disjoint : For all pairs A_i, A_j with $i \neq j$: $A_i \cap A_j = \emptyset$

Inclusion-Exclusion Principle (Siebformel)

For events $A_1, \dots, A_n$

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{1 \leq i_1 < i_2 \leq n} \Pr[A_{i_1} \cap A_{i_2}] + \cdots$$
$$+(-1)^{l+1} \sum_{1 \leq i_1 < \cdots < i_l \leq n} \Pr[A_{i_1} \cap \cdots \cap A_{i_l}] + \cdots$$
$$+(-1)^{n+1} \Pr[A_1 \cap \cdots \cap A_n] .$$

n=3:

Pr[A ∪ B ∪ C]
= Pr[A] + Pr[B] + Pr[C]
- Pr[A∩B] - Pr[A∩C] - Pr[B∩C] + Pr[A∩B∩C]

7

$$\Pr[\text{"A or B"}] = \Pr[A] + \Pr[B] = \underline{1}$$

$$\Pr[\text{"B or C"}] = \Pr[B \cup C] = \Pr[B] + \Pr[C] - \Pr[B \cap C]$$
$$= 3/6 + 2/6 - 1/6 = 4/6$$

$\Pr[\text{"A or B or C"}]$

$$= \Pr[A] + \Pr[B] + \Pr[C]$$
$$- \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C]$$
$$+ \Pr[A \cap B \cap C]$$

$$= \underline{1}$$

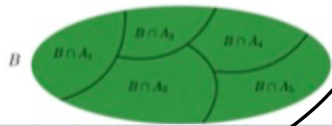
Conditional Probabilities

- **Definition:** If $\Pr[B] > 0$, then $\Pr[A|B] := \frac{\Pr[A \cap B]}{\Pr[B]}$.
- **Multiplication Rule:** If $\Pr[A_1 \cap \ldots \cap A_n] > 0$, then

$$\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdot \ldots \cdot \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B|A]$$
$$= \Pr[B] \cdot \Pr[A|B]$$

- **Law of Total Probability:**
  If $\Omega = A_1 \uplus \cdots \uplus A_n$ with $\Pr[A_1], \ldots, \Pr[A_n] > 0$, then
  $\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i]$.



2 Events: $\Pr[A] = \Pr[A|B] \cdot \Pr[B] + \Pr[A|\bar{B}] \cdot \Pr[\bar{B}]$

$$\Pr[A] = \Pr[A \cap B] + \Pr[A \cap \bar{B}]$$

- **Bayes' Theorem:**
  If $B \subseteq A_1 \uplus \cdots \uplus A_n$ with $\Pr[A_1], \ldots, \Pr[A_n], \Pr[B] > 0$, then

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$

$$\Pr[A|B] = \frac{\Pr[B|A] \cdot \Pr[A]}{\Pr[B]}$$

(1)

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \quad (2)$$

$$\Pr[A] \quad \Pr[B|A] = \frac{\Pr[A \cap B]}{\Pr[A]}$$

(3)

Probability Theory

Conditional Probability

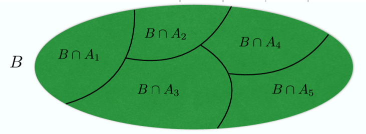
the probability that event A will occur if we already know that event B has occurred

- conditional probability :
  - Let A and B be arbitrary events with $Pr[B] > 0$. The conditional probability $Pr[A \mid B]$ of A given B is

$$Pr[A \mid B] := \frac{Pr[A \cap B]}{PR[B]}$$

$$Pr[A \cap B] = Pr[A \mid B] \cdot Pr[B]$$

$$Pr[A \cap B] = Pr[B \mid A] \cdot Pr[A]$$



Bedingte Wahrscheinlichkeiten

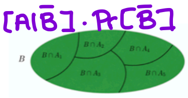
- **Definition:** Ist $\Pr[B] > 0$, so ist $\Pr[A|B] := \frac{\Pr[A \cap B]}{\Pr[B]}$.
- **Multiplikationssatz:** Ist $\Pr[A_1 \cap \ldots \cap A_n] > 0$, so ist

$$\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdot \ldots \cdot \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

$Pr[A \cap B] = Pr[A] \cdot P[B \mid A]$

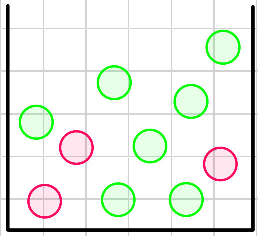
- **Satz von der totalen Wahrscheinlichkeit:** Ist $\Omega = A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n] > 0$, so gilt $\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i]$.

$Pr[A] = Pr[A \mid B] \cdot Pr[B] + Pr[A \mid \bar{B}] \cdot Pr[\bar{B}]$



- **Satz von Bayes:** Ist $B \subseteq A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n], \Pr[B] > 0$, so gilt

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$



7 green

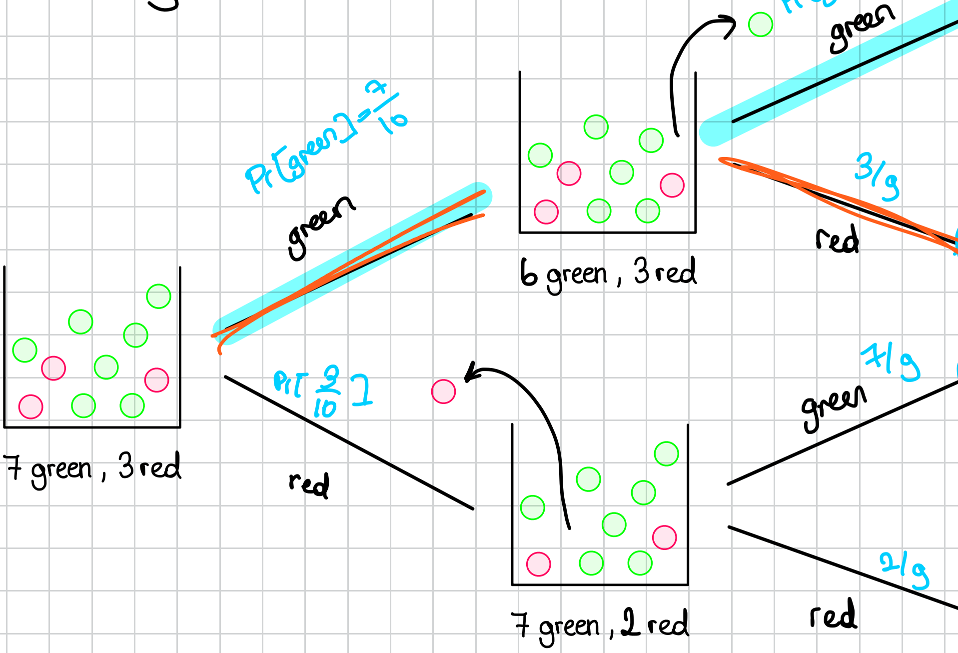
3 red

Pick 2 without putting back.

1) What is the probability of picking green and red afterwards?

2) What is the probability of picking only 1 green ball?

"Baumdiagramm"



$Pr[green \cap green]$ $\frac{7}{10} \cdot \frac{6}{9}$

$Pr[green \cap red]$ $\boxed{\frac{7}{10} \cdot \frac{3}{9}}$

$Pr[red \cap green]$ $\frac{3}{10} \cdot \frac{7}{9}$

$Pr[red \cap red]$ $\frac{3}{10} \cdot \frac{2}{9}$

Probability Theory
Conditional Probability

the probability that event A will occur if we already know that event B has occurred

- conditional probability :
  - Let A and B be arbitrary events with $Pr[B] > 0$. The conditional probability $Pr[A \mid B]$ of A given B is

$$Pr[A \mid B] := \frac{Pr[A \cap B]}{PR[B]}$$

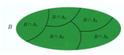
$$Pr[A \cap B] = Pr[A \mid B] \cdot Pr[B]$$
$$Pr[A \cap B] = Pr[B \mid A] \cdot Pr[A]$$

Bedingte Wahrscheinlichkeiten

- **Definition:** Ist $\Pr[B] > 0$, so ist $\Pr[A|B] := \frac{\Pr[A \cap B]}{\Pr[B]}$.
- **Multiplikationssatz:** Ist $\Pr[A_1 \cap \ldots \cap A_n] > 0$, so ist

$$\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdot \ldots \cdot \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

- **Satz von der totalen Wahrscheinlichkeit:**
  Ist $\Omega = A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n] > 0$, so gilt $\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i]$.

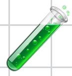


- **Satz von Bayes:**
  Ist $B \subseteq A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n], \Pr[B] > 0$, so gilt

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$

Imagine a fellow student of yours is stopped by the police after a night out and has to take a breathalyzer test.

The test detects alcohol consumption in 99.9% of cases.

However, it also produces a positive result in 3% of cases, even though the person being tested hasn't consumed any alcohol. 

We also know that 5% of the people tested have actually consumed alcohol: 

The test is positive for your fellow student. What is the probability that they have actually consumed alcohol?

+ / − alcohol / no alcohol

$$Pr[+ \mid alcohol] = 99.9/100$$

$$Pr[+ \mid no\ alcohol] = 3/100$$

$$Pr[alcohol] = 5/100$$

$$Pr[alcohol \mid +] = ?$$

Bayes

$$Pr[alcohol \mid +] = \frac{Pr[+ \mid alcohol] \cdot Pr[alcohol]}{Pr[+]}$$

$$Pr[+] = Pr[+ \mid alcohol] \cdot Pr[alcohol] + Pr[+ \mid no\ alcohol] \cdot Pr[no\ alcohol]$$

1 − alcohol

Independence

- **Definition:** $X_1, \dots, X_n$ are independent if and only if for all $(x_1, \dots, x_n) \in W_{X_1} \times \dots \times W_{X_n}$ holds: $\Pr[X_1 = x_1, \dots, X_n = x_n] = \Pr[X_1 = x_1] \cdot \ldots \cdot \Pr[X_n = x_n]$.
- **Multiplication Formula:** If $X_1, \dots, X_n$ are independent and $S_i \subseteq W_{X_i}$, then $\Pr[X_1 \in S_1, \dots, X_n \in S_n] = \Pr[X_1 \in S_1] \cdot \ldots \cdot \Pr[X_n \in S_n]$.

Roll 2 dices X_1 X_2

$\Pr[X_1 = 3, X_2 = 5] = 1/6 \cdot 1/6$

$S_1 = \{1, 2, 3\}$ "outcome low"

$S_2 = \{5, 6\}$ outcome high

$$\Pr[X_1 \in S_1, X_2 \in S_2] = \Pr[X_1 \in S_1] \cdot \Pr[X_2 \in S_2]$$
$$= 3/6 \cdot 2/6$$

(c) Oliver owns three pairs of shoes – two blue pairs, and one yellow, which he stores unordered in his wardrobe. One morning, during a power outage, he has to put on his shoes in complete darkness. He randomly (uniformly at random) grabs two shoes from the wardrobe and tries to put them on.

We let A denote the event that he picked one left shoe and one right shoe (i.e. he is able to put on the shoes he picked), and we let B be the event that the two shoes he picked have the same color.

Model this setting as a probability space and compute $\Pr[A]$ and $\Pr[A|B]$.