A&W

Exercise Session 8

Probability III

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A&W Overview

A&W

Connectivity
- Articulation Points
- Bridges
- Block - Decomposition
- Menger's Theorem

Cycles
- Eulerian Cycle
- Hamiltonian Cycle
- TSP

Matchings
- Definition
- Algorithms
- Hall's Theorem

Colorings
- Definition
- Algorithm
- Brooks's Theorem

Wahrscheinlichkeit
- Grundbegriffe und Notationen
- Bedingte Wahrscheinlichkeiten
- Unabhängigkeiten
- Zufallsvariablen
- Wichtige Diskrete Verteilungen
- Abschätzen von Wahrscheinlichkeiten

Randomized Algorithms
- Las-Vegas
- Monte-Carlo
- Primality Test
- Target - Shooting
- Finding Duplicates
- Longest Path Problem

Graph Algorithms

Flow
- Definition
- Maxflow - Mincut
- Ford - Fulkerson
- Matching w. Flow
- Edge-disjoint paths w. Flow

Minimum Cut
- Definition
- Cut(G) Algorithm
- Bootstrapping

Geometric Algorithms

Convex Hull
- Definition
- Jarvis Wrap
- Local optimization

Smallest Enclosing Circle
- Definition
- First Algorithm
- Final Algorithm

Outline

- Minitest 4

- Probability Theory III
  - independence of random variables
  - wald’s identity
  - inequalities

- A Game of Skill - Probability CodeExpert

Minitest 4

Probability Theory

Probability Theory

Independence of random variables

- independent (random variables)
  - $X_1, \ldots, X_n$ are independent if for all $x_1 \in X_{X_1}, , , x_n \in W_{X_n}$ the events $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ are independent.

$$\underbrace{\Pr[X_1 = x_1, \ldots, X_n = x_n]}_{=f_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)} = \underbrace{\Pr[X_1 = x_1]}_{=f_{X_1}(x_1)} \cdots \underbrace{\Pr[X_n = x_n]}_{=f_{X_n}(x_n)}$$

Aber: Viele der Gleichungen sind *redundant*.

Beispiel für n = 3: Falls X_3 Wertebereich {0,1} hat, dann folgt aus

$$\Pr[X_1 = x_1, X_2 = x_2, X_3 = 0] = \Pr[X_1 = x_1] \cdot \Pr[X_2 = x_2] \cdot \Pr[X_3 = 0], \text{ und}$$
$$\Pr[X_1 = x_1, X_2 = x_2, X_3 = 1] = \Pr[X_1 = x_1] \cdot \Pr[X_2 = x_2] \cdot \Pr[X_3 = 1]$$

automatisch:

$$\begin{aligned}\Pr[X_1 = x_1, X_2 = x_2] &= \Pr[X_1 = x_1, X_2 = x_2, X_3 = 0] + \Pr[X_1 = x_1, X_2 = x_2, X_3 = 1] \\ &= \Pr[X_1 = x_1] \cdot \Pr[X_2 = x_2] \cdot \Pr[X_3 = 0] + \Pr[X_1 = x_1] \cdot \Pr[X_2 = x_2] \cdot \Pr[X_3 = 1] \\ &= \Pr[X_1 = x_1] \cdot \Pr[X_2 = x_2] \cdot (\Pr[X_3 = 0] + \Pr[X_3 = 1]) \\ &= \Pr[X_1 = x_1] \cdot \Pr[X_2 = x_2].\end{aligned}$$

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Probability Theory
Independence of random variables

- independent (random variables)
  - $X_1, \ldots, X_n$ are independent if for all $x_1 \in X_{X_1}, , , x_n \in W_{X_n}$ the events $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ are independent.

$$\underbrace{\Pr[X_1 = x_1, \ldots, X_n = x_n]}_{=f_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)} = \underbrace{\Pr[X_1 = x_1]}_{=f_{X_1}(x_1)} \cdot \ldots \cdot \underbrace{\Pr[X_n = x_n]}_{=f_{X_n}(x_n)},$$

- $X_1, \ldots, X_n$ are independent random variables, $S_1, \ldots, S_n \subseteq \mathbb{R}$ arbitrary :
  - $\Pr[X_1 \in S_1, \ldots, X_n \in S_n] = \Pr[X_1 \in S_1] \cdots \Pr[X_n \in S_n]$.
- $X_1, \ldots, X_n$ are independent random variables , $f_1, \ldots, f_n$ are real-valued functions ($f_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, n$) :
  - $f_1(X_1), \ldots, f_n(X_n)$ are independent random variables

Probability Theory
Independence of random variables

- independent (random variables)
  - $X_1, \ldots, X_n$ are independent if for all $x_1 \in X_{X_1}, , , x_n \in W_{X_n}$ the events $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ are independent.

$$\underbrace{\Pr[X_1 = x_1, \ldots, X_n = x_n]}_{=f_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)} = \underbrace{\Pr[X_1 = x_1]}_{=f_{X_1}(x_1)} \cdot \ldots \cdot \underbrace{\Pr[X_n = x_n]}_{=f_{X_n}(x_n)},$$

- X and Y are two independent random variables, $Z := X + Y$

$$f_Z(z) = \sum_{x \in W_X} f_X(x) \cdot f_Y(z - x)\,.$$

$$\text{Poisson}(\lambda_1) + \text{Poisson}(\lambda_2) = \text{Poisson}(\lambda_1 + \lambda_2)$$

$$\text{Bin}(n, p) + \text{Bin}(m, p) = \text{Bin}(n + m, p)$$

Probability Theory
Wald's Identity

- N and X are two independent random variables, $W_N \subseteq \mathbb{N}$

$$Z := \sum_{i=1}^{N} X_i \quad \text{where } X_1, X_2 \ldots \text{ are independent copies of } X$$

$$\mathbb{E}[Z] = \mathbb{E}[N] \cdot \mathbb{E}[X]$$

Example

Probability Theory
Inequalities

get used to using it !

Abschätzungen

- **Boolesche Ungleichung, Union Bound:** $\Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \Pr[A_i]$.
- **Markov:** Ist $W_X \subseteq \mathbb{R}_{\geq 0}$ und $t \in \mathbb{R}_{\geq 0}$, so ist $\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$ bzw. $\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t}$.
- **Chebyshev:** Für $t \in \mathbb{R}_{\geq 0}$ ist $\Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$ bzw. $\Pr[|X - \mathbb{E}[X]| \geq t \cdot \sigma[X]] \leq \frac{1}{t^2}$.
- **Chernoff:** Seien $X_1, \ldots, X_n$ unabhängig und Bernoulli-verteilt, $X := \sum_{i=1}^{n} X_i$ und $\delta \in [0, 1]$. Dann ist

$$\Pr[X \geq (1 + \delta)\mathbb{E}[X]] \leq e^{-\frac{1}{3}\delta^2 \mathbb{E}[X]},$$

$$\Pr[X \leq (1 - \delta)\mathbb{E}[X]] \leq e^{-\frac{1}{2}\delta^2 \mathbb{E}[X]},$$

$$\Pr[X \geq t] \leq 2^{-t} \qquad \text{für } t \geq 2e\mathbb{E}[X].$$

Let's take a break

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Questions

Feedbacks , Recommendations

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