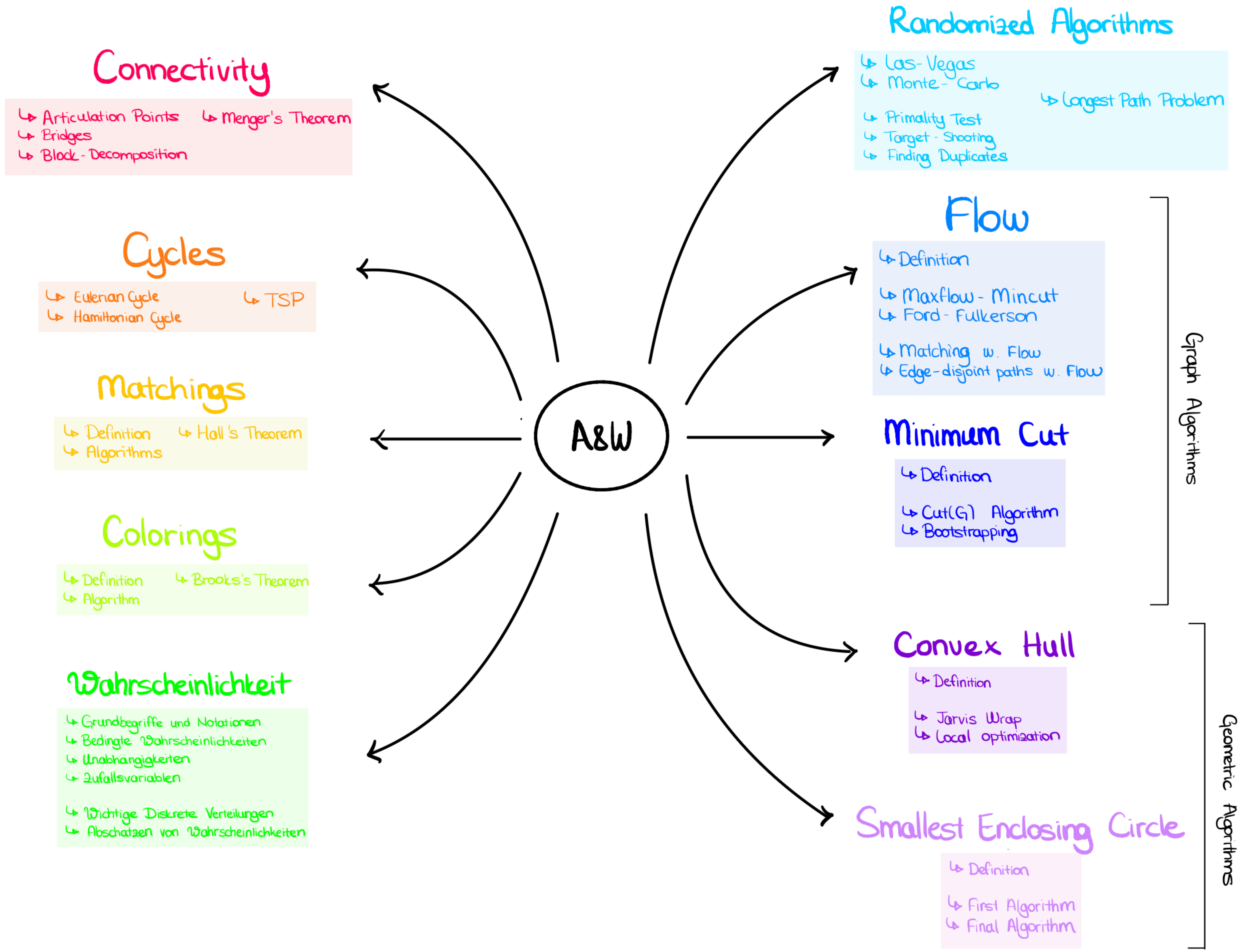
A&W

Exercise Session 7

Probability II

Nil Ozer

A&W Overview



A&W
Connectivity
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↳ Block-Decomposition
↳ Menger's Theorem
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↳ Hamiltonian Cycle
↳ TSP
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↳ Local optimization
Smallest Enclosing Circle
↳ Definition
↳ First Algorithm
↳ Final Algorithm
Geometric Algorithms

Outline

- Coloring Kahoot

- Probability Theory II
  - Conditional independence
  - Random variables
  - Expected value , Variance
  - Distributions

Coloring Kahoot

Let's take a break

A&W 😊🧑‍💻
WhatsApp group

Probability Theory

Probability Theory
Conditional Independence

- conditional independence :
  - Event A and B are independent , if

$$Pr[A \cap B] = Pr[A] \cdot Pr[B]$$

  - Events A ,B and C are independent , if

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$$

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$
$$\Pr[A \cap C] = \Pr[A] \cdot \Pr[C]$$
$$\Pr[B \cap C] = \Pr[B] \cdot \Pr[C]$$

Probability Theory

Conditional Independence - Formelsammlung

Unabhängigkeit

- **Definition:** $X_1, \ldots, X_n$ heissen genau dann unabhängig, wenn für alle $(x_1, \ldots, x_n) \in W_{X_1} \times \ldots \times W_{X_n}$ gilt: $\Pr[X_1 = x_1, \ldots, X_n = x_n] = \Pr[X_1 = x_1] \cdot \ldots \cdot \Pr[X_n = x_n]$.
- **Multiplikationsformel:** Sind $X_1, \ldots, X_n$ unabhängig und $S_i \subseteq W_{X_i}$, dann gilt: $\Pr[X_1 \in S_1, \ldots, X_n \in S_n] = \Pr[X_1 \in S_1] \cdot \ldots \cdot \Pr[X_n \in S_n]$.
- **Transformationen:** Seien $f_i : \mathbb{R} \to \mathbb{R}$. Wenn $X_1, \ldots, X_n$ unabhängig sind, dann gilt dies auch für $f(X_1), \ldots, f(X_n)$.
- **Summe:** Sind X, Y unabhängig und $Z := X + Y$, so gilt $f_Z(z) = \sum_{x \in W_X} f_X(x) \cdot f_Y(z - x)$.

Probability Theory

Random Variable

- random variable X :
  - A random variable X is a measurable function $X:\Omega \to \mathbb{R}$ from a sample space Ω as a set of possible outcomes to real numbers

Random process

Outcomes → Numbers

Flipping a coin

$$X = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails} \end{cases}$$

Rolling 8 dices

$Y =$ Sum of the upward faces after rolling 8 dices

Probability Theory
Random Variable

- random variable X : Random process **Outcomes → Numbers**
    - A random variable X is a measurable function $X : \Omega \to \mathbb{R}$ from a sample space Ω as a set of possible outcomes to real numbers

Flipping a coin

$$X = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails} \end{cases}$$

$$Pr[X = 1] = Pr[w \in \Omega \,|\, X(w) = 1]$$

Rolling 8 dices

$Y =$ Sum of the upward faces after rolling 8 dices

$$Pr[Y \leq 6] = Pr[w \in \Omega \,|\, X(w) \leq 6]$$

Probability Theory
Functions

- probability density function $f_x(x)$
  - $f_X : \mathbb{R} \to [0,1], \quad x \mapsto \Pr[X = x] \quad (= \Pr[X(\omega) = x])$

- cumulative distribution function $F_x(x)$
  - $F_X : \mathbb{R} \to [0,1], \quad x \mapsto \Pr[X \leq x]$

Probability Theory

Expected Value

- expected value $\mathbb{E}[X]$
  - $$\mathbb{E}[X] := \sum_{x \in W_x} x \cdot Pr[X = x]$$
  - $$\mathbb{E}[X] = \sum_{w \in \Omega} X(w) \cdot Pr[w]$$
  - $$\mathbb{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$$

weighted average

Probability Theory
Expected Value Properties

- expected value
  - $\mathbb{E}[X] = \sum_{w \in \Omega} X(w) \cdot Pr[w]$

weighted average

- linearity:
  - $E[X + Y] = E[X] + E[Y]$
  - $E[aX] = aE[x]$
  - $E[a_1X_1 + a_2X_2 + \ldots + a_nX_N + b] = a_1E[X_1] + \ldots + a_nE[X_n] + b$
- monotonicity :
  - If $X \leq Y$ then $E[X] \leq E[Y]$

Probability Theory
Variance

- variance $Var[X]$

$$\mathrm{Var}[X] := \mathbb{E}[(X - E[x])^2] = \sum_{x \in W_X} (x - E[x])^2 \cdot \Pr[X = x]$$

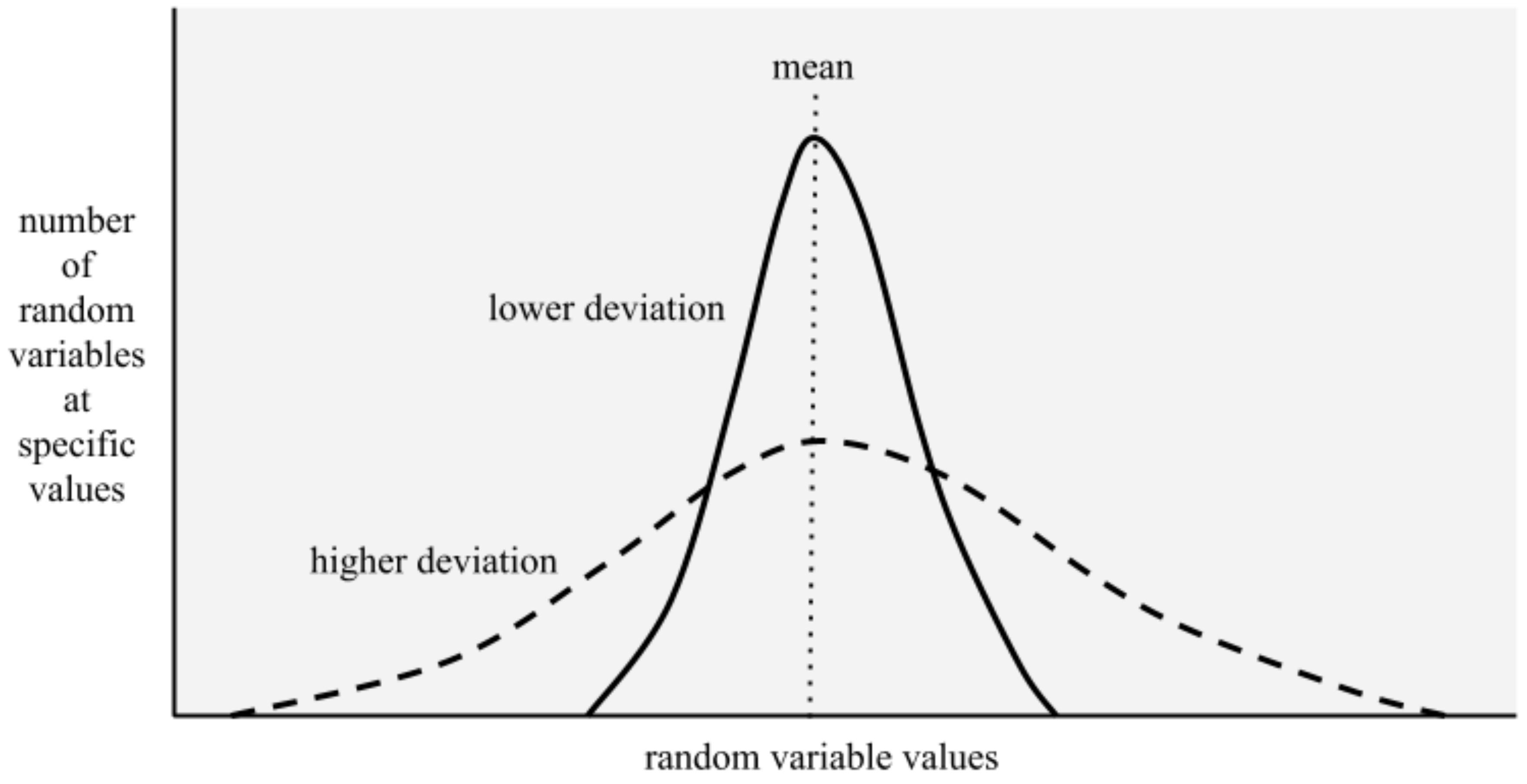
$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathrm{Var}[a \cdot X + b] = a^2 \cdot \mathrm{Var}[X]$$

- standard deviation σ

$$\sigma := \sqrt{\mathrm{Var}[X]}$$

measure of how far a set of numbers is spread out from their average value/mean



Probability Theory

Formelsammlung

Erwartungswert

- **Definition:** $\mathbb{E}[X] := \sum_{x \in W_X} x \cdot \Pr[X = x]$
- **Linearität:** Für $a_1, \ldots, a_n, b \in \mathbb{R}$ gilt $\mathbb{E}[a_1 X_1 + \ldots + a_n X_n + b] = a_1 \mathbb{E}[X_1] + \ldots + a_n \mathbb{E}[X_n] + b$.
- **Summenformel:** Ist $W_X \subseteq \mathbb{N}_0$, dann gilt $\mathbb{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$.
- **Multiplikativität:** Für unabhängige $X_1, \ldots, X_n$ gilt $\mathbb{E}[X_1 \cdot \ldots \cdot X_n] = \mathbb{E}[X_1] \cdot \ldots \cdot \mathbb{E}[X_n]$.

Varianz

- **Definition:** $\mathrm{Var}[X] := \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
- **Translation:** Für $a, b \in \mathbb{R}$ gilt $\mathrm{Var}[a \cdot X + b] = a^2 \cdot \mathrm{Var}[X]$.
- **Standardabweichung:** $\sigma[X] := \sqrt{\mathrm{Var}[X]}$.
- **Additivität:** Für unabhängige $X_1, \ldots, X_n$ gilt $\mathrm{Var}[X_1 + \ldots + X_n] = \mathrm{Var}[X_1] + \ldots + \mathrm{Var}[X_n]$.

Probability Theory

Distributions

Probability Theory

Indicator Variable

- indicator variable I_A:
  - For an event $A \subseteq \Omega$

$$I_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

  - $p = Pr[A] = E[I_A]$

$$f_{I_A}(x) = \begin{cases} p, & x = 1, \\ 1 - p, & x = 0, \\ 0, & \text{otherwise} \end{cases}$$

Probability Theory
Bernoulli Distribution

$$X \sim \text{Bernoulli}(p)$$

$$f_X(x) = \begin{cases} p, & x = 1, \\ 1 - p, & x = 0, \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = p \qquad Var[X] = p\,(1 - p)$$

yes-no question

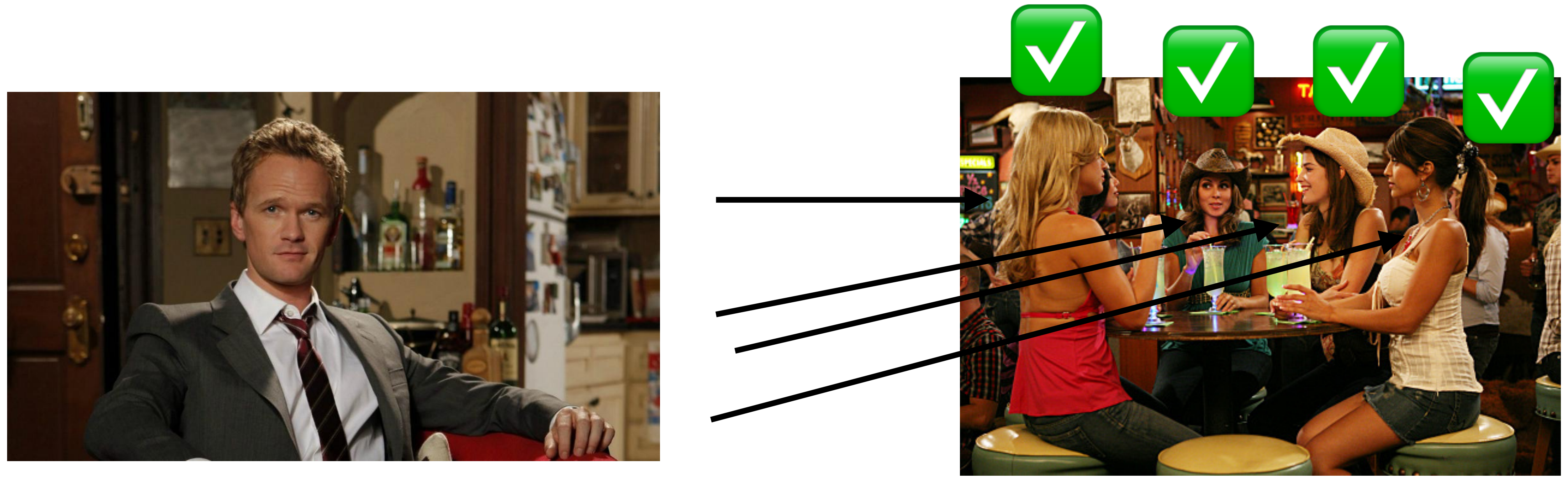
Example to remember :

Coin toss

X = indicator for head

Probability Theory

Binomial Distribution



$$X \sim \mathrm{Bin}(n, p)$$

sucesses in a sequence of n yes-no questions

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, \ldots, n\} \\ 0, & \text{otherwise} \end{cases}$$

Example to remember :

Coin toss 10 times

X = #heads

$$E[X] = np \qquad Var[X] = np\,(1-p)$$

Probability Theory

Poisson-Distribution

$$X \sim \text{Po}(\lambda)$$

$$f_X(i) = \begin{cases} \frac{e^{-\lambda}\lambda^i}{i!}, & \text{für } i \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

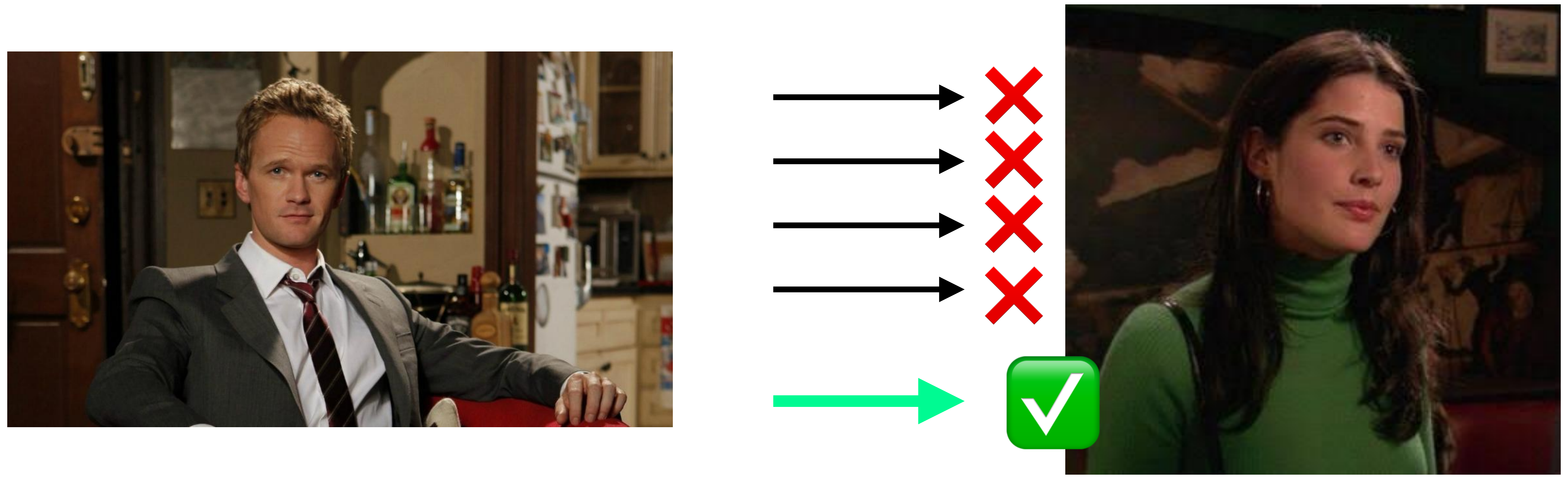
$\text{Bin}(n, \lambda/n)$ converges to $\text{Po}(\lambda)$

for $n \to \infty$

$$E[X] = \lambda \qquad Var[X] = \lambda$$

Probability Theory

Geometric Distribution



$$X \sim \text{Geo}(p)$$

$$f_X(i) = \begin{cases} p(1-p)^{i-1}, & \text{für } i \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(n) = 1 - (1-p)^n$$

$$E[X] = 1/p \qquad Var[X] = \frac{1-p}{p^2}$$

#yes-no questions needed to get one yes

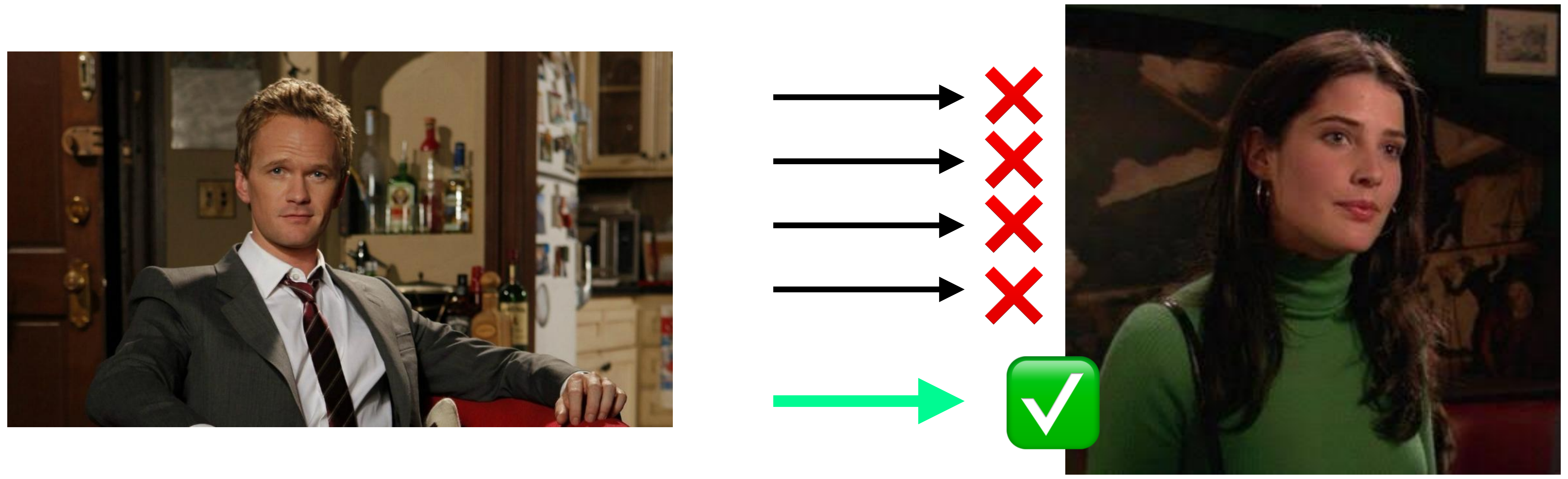
Example to remember :

Coin toss until a head comes

X = #tosses

Probability Theory

Geometric Distribution



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$$F_X(n) = 1 - (1-p)^n$$

$$E[X] = 1/p \qquad Var[X] = \frac{1-p}{p^2}$$

Robin has no brain

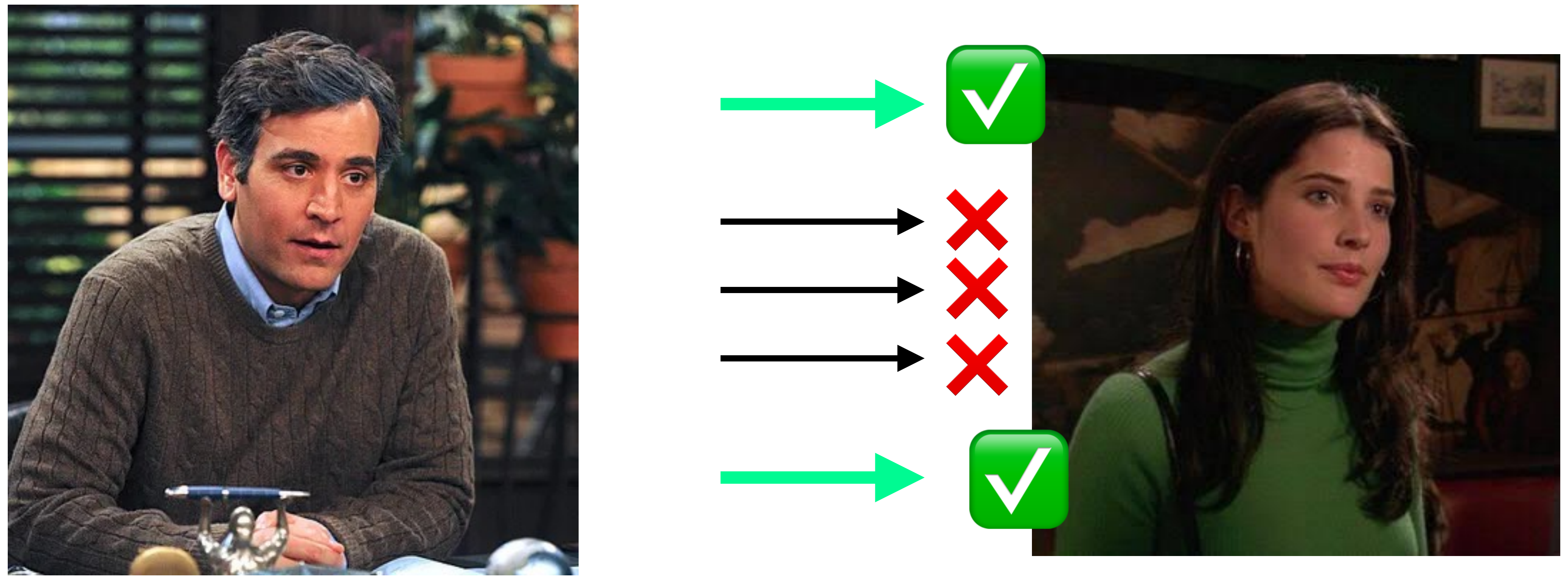
Memorylessness

$$\Pr[X \geq s + t \mid X > s] = \Pr[X \geq t]$$

$$\Pr[X = s + t \mid X > s] = \Pr[X = t]$$

Probability Theory

Negative Binomial Distribution



$$X \sim \text{NegativeBinomial}(n)$$

#yes-no questions needed to get n yesses

$$f_X(k) = \begin{cases} \binom{k-1}{n-1}(1-p)^{k-n}p^n, & \text{for } k = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

Example to remember :

Coin toss until n-th head comes

X = #tosses

$$E[X] = n/p$$

Probability Theory

Coupon Collector

$$\underbrace{2}_{1},\ \underbrace{2,1}_{2},\ \underbrace{2,2,3}_{3},\ \underbrace{1,3,2,3,1,4}_{4}$$

erhaltenes Bild	2	2, 1	2, 2, 3	1, 3, 2, 3, 1, 4
Phase	1	2	3	4
X_i	1	2	3	6

phase i := turns while we have i-1 different coupons

$X_i :=$ #turns in phase i

$$X_i \sim \text{Geo}\left(\frac{n-(i-1)}{n}\right) \quad E[X_i] = 1/p$$

$$X = \sum_{i=1}^{n} X_i$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \cdot \sum_{i=1}^{n} \frac{1}{i} = n \cdot H_n,$$

$H_n = \ln n + \mathcal{O}(1)$

collect all coupons and win

Example to remember :

n different coupons , we're getting one in each turn

X = #turns until we get all n coupons

Probability Theory

Formelsammlung

Wichtige Verteilungen

Name	Bezeichnung	Wertebereich	Dichte	Erwartungswert	Varianz
Bernoulli	Bernoulli(p)	$\{0,1\}$	$f_X(i) = \begin{cases} p & \text{für } i=1, \\ 1-p & \text{für } i=0. \end{cases}$	p	$p(1-p)$
Binomial	Bin(n,p)	$\{0,1,\dots,n\}$	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometrisch	Geo(p)	$\mathbb{N}$	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	$\mathbb{N}_0$	$f_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	λ	λ

Examples

Questions

Feedbacks , Recommendations

Nil Ozer