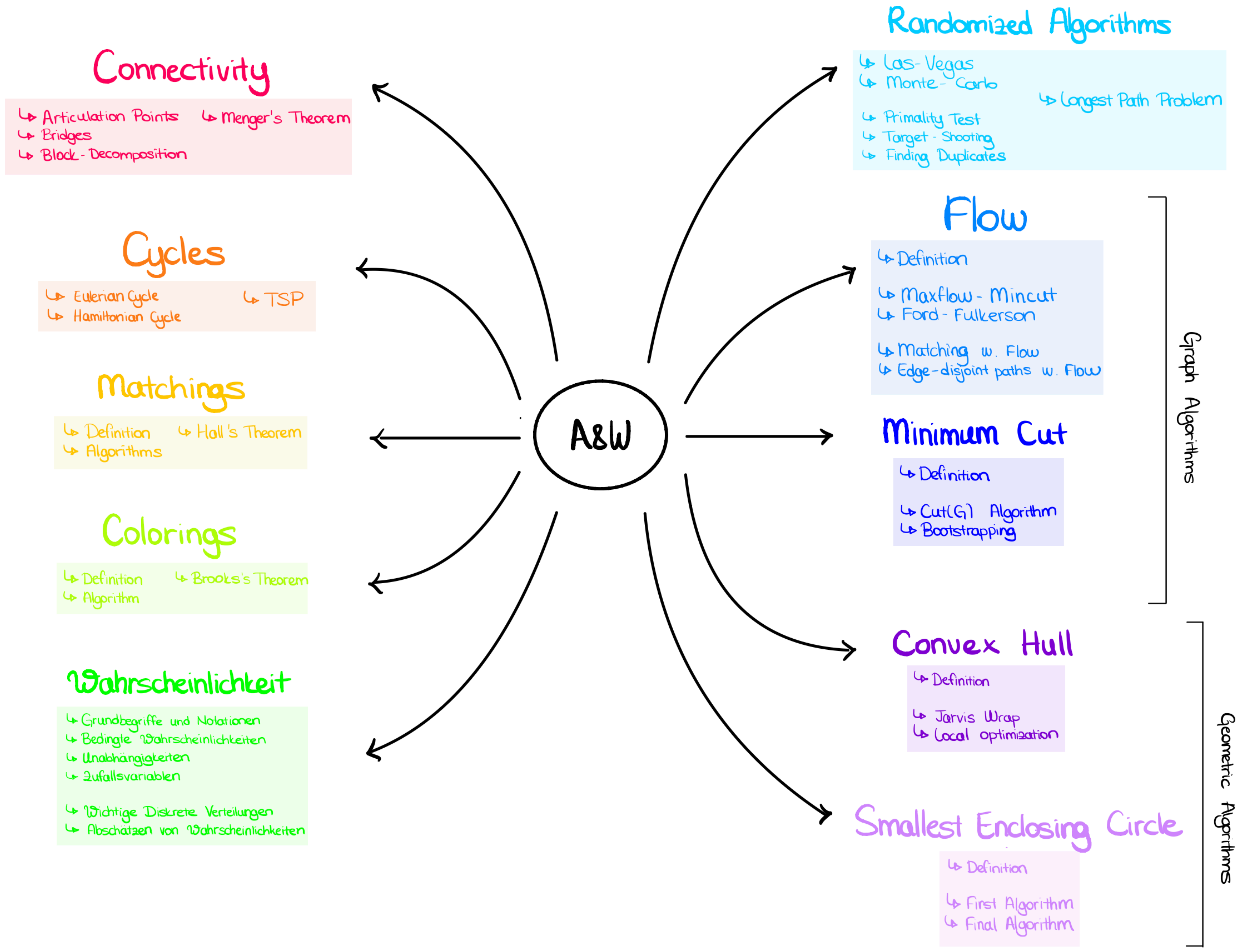
A&W

Exercise Session 12

Minimum Cut, Smallest Enclosing Cycle

Nil Ozer

A&W Overview



A&W
Connectivity
Articulation Points
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Block - Decomposition
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Eulerian Cycle
Hamiltonian Cycle
TSP
Matchings
Definition
Algorithms
Hall's Theorem
Colorings
Definition
Algorithm
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Grundbegriffe und Notationen
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Unabhängigkeiten
Zufallsvariablen
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Abschätzen von Wahrscheinlichkeiten
Randomized Algorithms
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Flow
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Maxflow - Mincut
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Minimum Cut
Definition
Cut(G) Algorithm
Bootstrapping
Graph Algorithms
Convex Hull
Definition
Jarvis Wrap
Local optimization
Smallest Enclosing Circle
Definition
First Algorithm
Final Algorithm
Geometric Algorithms

Last Weeks …

- 08.05 : Randomized Algorithms II
- 15.05 : Flow
- 23.05 online : Minimum Cut , Smallest Enclosing Circle
- 28.05 extra session : Exam Prep Session + Pizza and Drinks
- 30.05 last extra session : Convex Hull (shortly remaining primality tests)

Outline

- Minimum Cut
- Smallest Enclosing Circle

Minimum Cut

Min-Cut

Definitions

- Multigraph :
  - undirected, unweighted, without self-loops
  - possibly with multiple edges between the same pair of nodes

Min-Cut

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- Edge Cut C :
  - A set of edges C s.t. $G' = (V, E \backslash C)$ is a disconnected graph.

Min-Cut
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- Edge Cut C :
  - A set of edges C s.t. $G' = (V, E \backslash C)$ is a disconnected graph.
- $\mu(G)$:
  - the cardinality of the smallest possible edge cut in graph G.

$$\mu(G) := \min_{\substack{C \subseteq E, \\ (V, E \backslash C) \text{ disconnected}}} |C|$$

Min-Cut

Problem Description

given : A multigraph G

to find : $\mu(G)$

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Min-Cut

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Examples :

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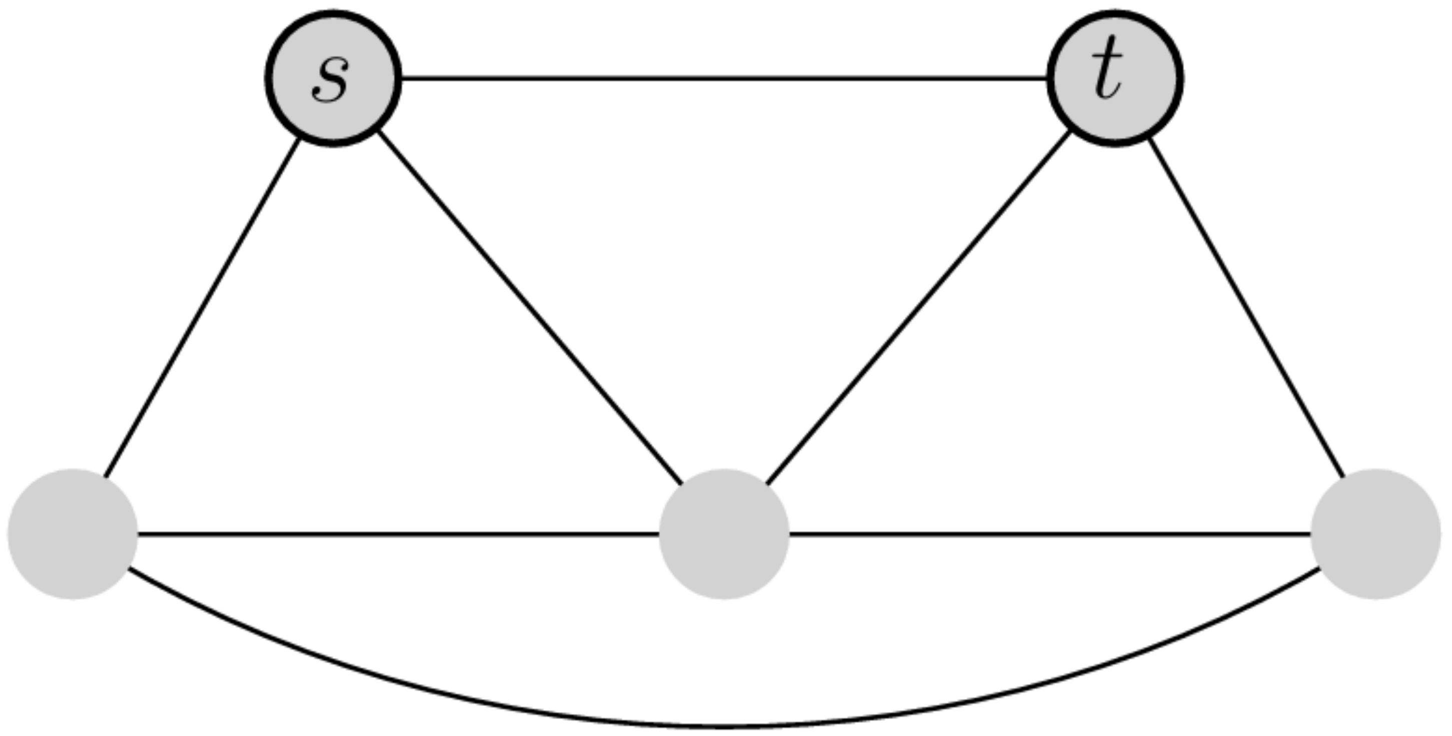
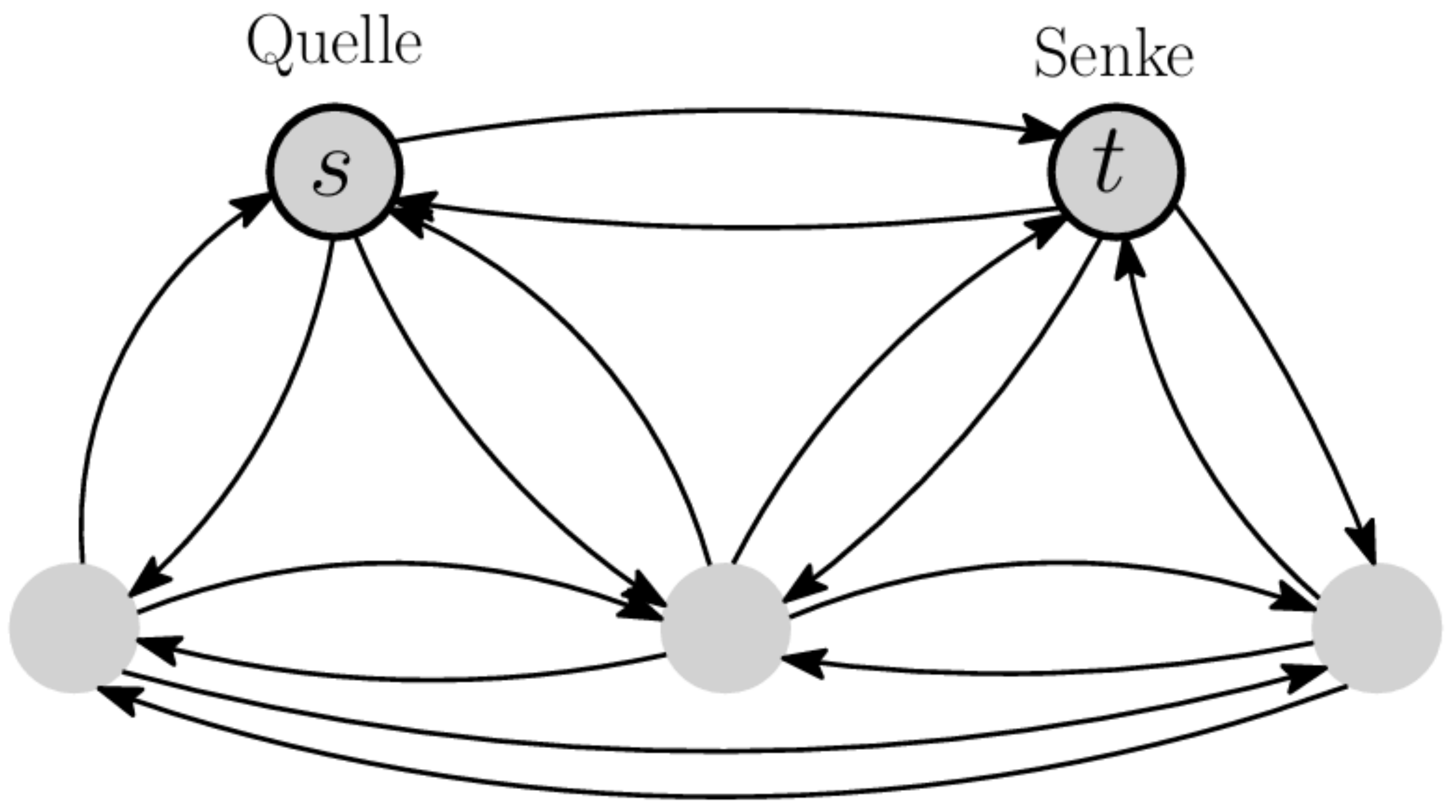
Min-Cut
First Known Solution

minimum s-t cut in O(mn logn)





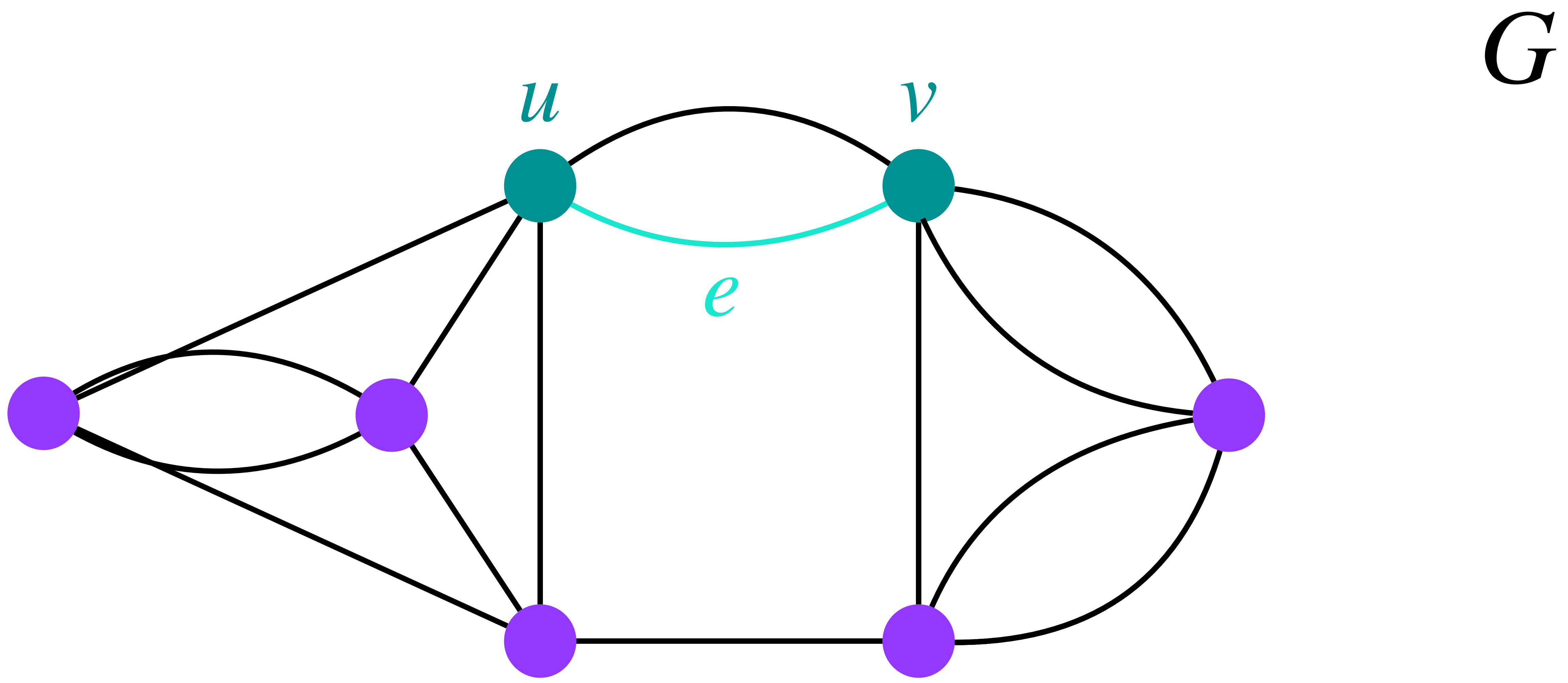
- Fix a source node s
- Then consider $t \in V \setminus \{s\}$, for each t compute the minimum s-t cut
- The global min-cut is the smallest of these s-t cuts

total runtime : $O((n-1)mn \log n) = O(n^4 \log n)$

Min-Cut

Edge Contraction of e

$e = \{u, v\} \in E$



Min-Cut

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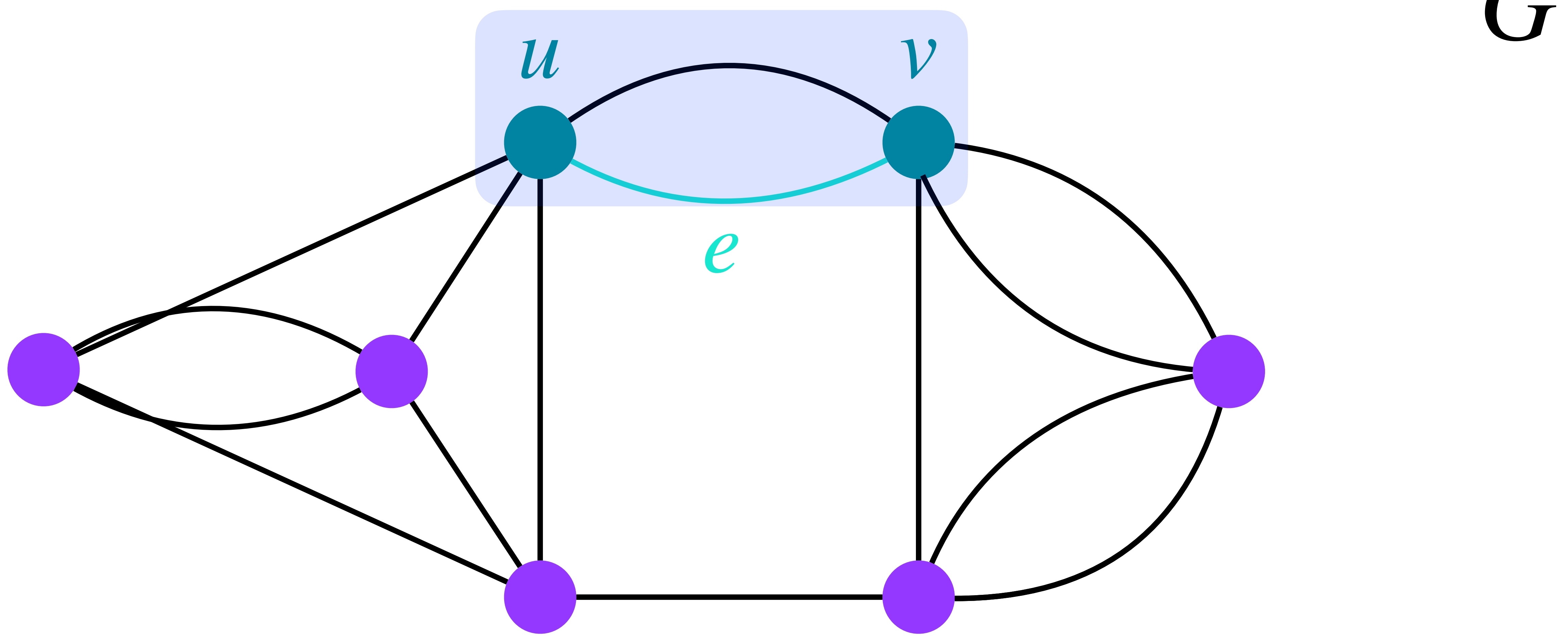

For $w, w' \in V(G) \setminus \{u, v\}$:

$$\{w, w'\} \mapsto \{w, w'\}, \quad \{w, u\} \mapsto \{w, x_{u,v}\}, \quad \{w, v\} \mapsto \{w, x_{u,v}\}$$

Min-Cut

Edge Contraction of e

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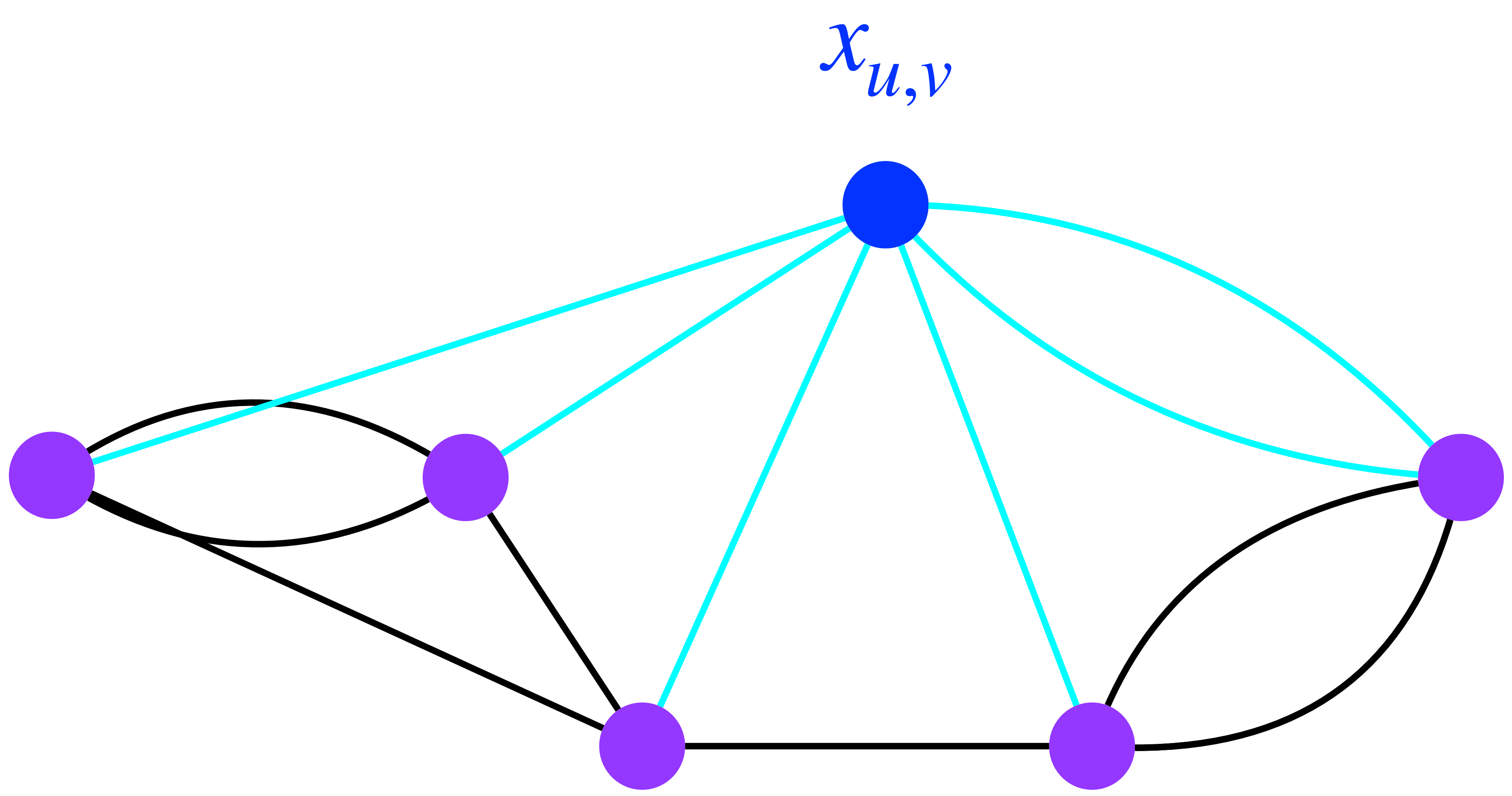

G / e

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$$\{w, w'\} \mapsto \{w, w'\}, \quad \{w, u\} \mapsto \{w, x_{u,v}\}, \quad \{w, v\} \mapsto \{w, x_{u,v}\}$$

Min-Cut

Lemma

- $\mu(G)$:
  - the cardinality of the smallest possible edge cut in graph G.

Let $G = (V, E)$ be a multigraph, $e \in E$

$$\mu(G \backslash e) \geq \mu(G)$$

If G has a minimum cut C s.t. $e \notin C$

$$\mu(G \backslash e) = \mu(G)$$

Find e whose contraction preserves μ

The minimum cut value μ can never decrease when contracting an edge

μ stays unchanged if there exists a minimum cut that doesn't contain the edge being contracted

Min-Cut

Cut(G)

```
G' ← G
while |V(G')| > 2 do
    e ← uniformly random edge in G'
    G' ← G'\e
return size of the unique cut in G'
```

Runtime : $O(n^2)$

Min-Cut

Cut(G)

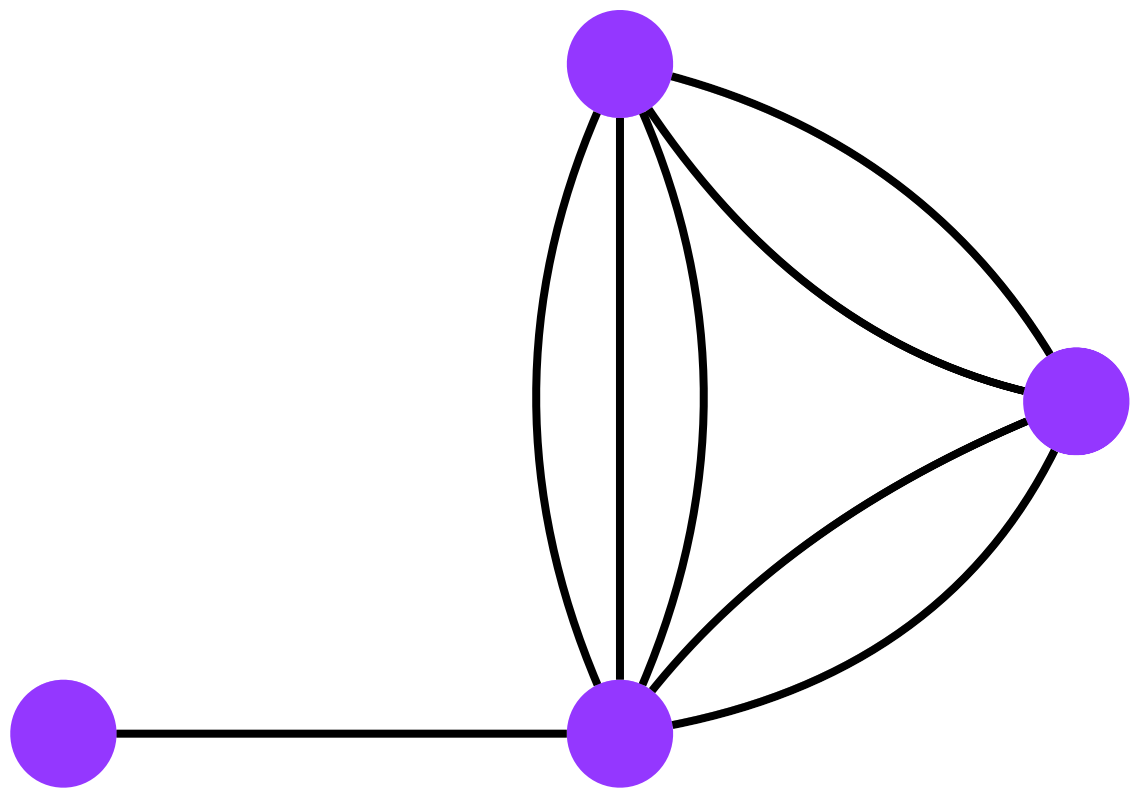
$G' \leftarrow G$

while $|V(G')| > 2$ do

    $e \leftarrow$ uniformly random edge in G'

    $G' \leftarrow G' \backslash e$

return size of the unique cut in G'



Min-Cut

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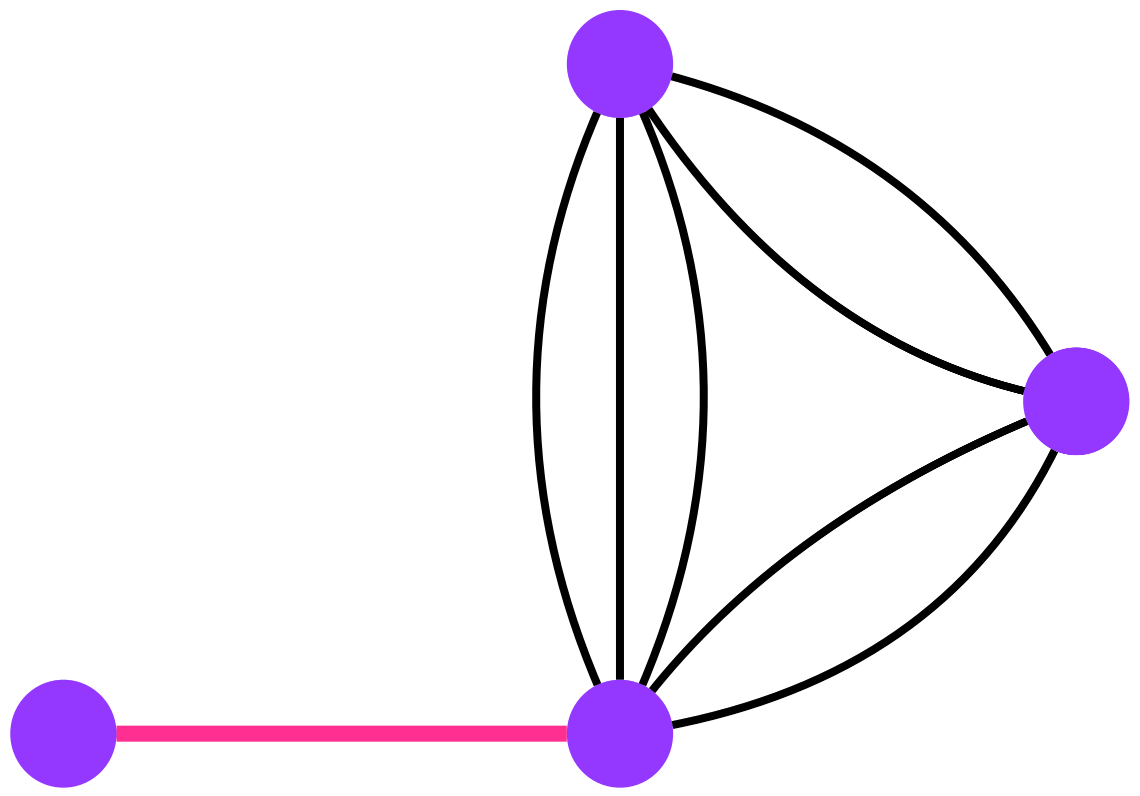

Min-Cut

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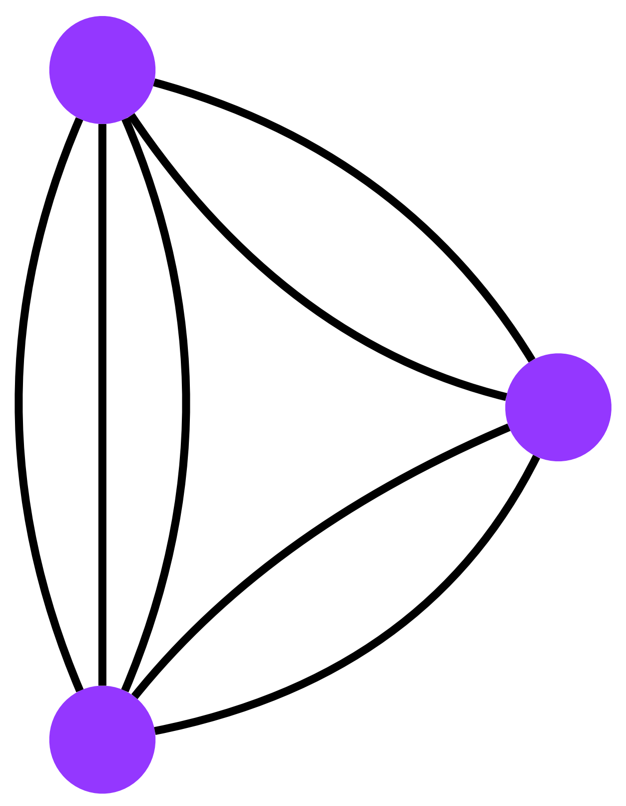

Min-Cut

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$G' \leftarrow G$

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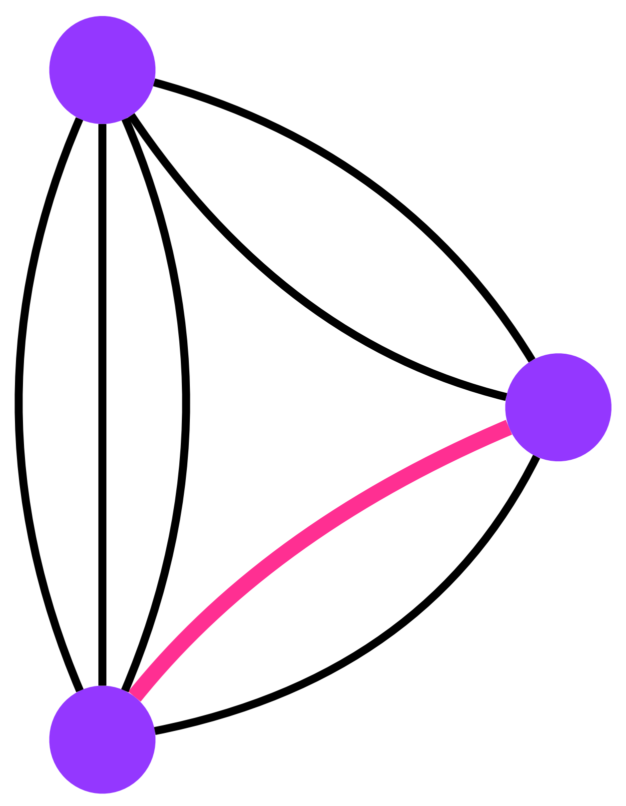
    $e \leftarrow$ uniformly random edge in G'

    $G' \leftarrow G' \setminus e$

return size of the unique cut in G'



Min-Cut
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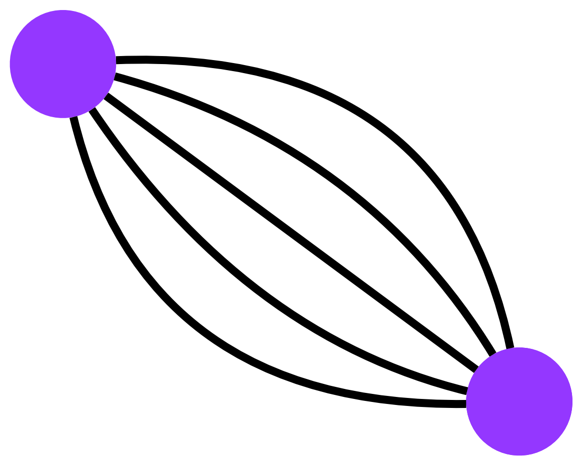
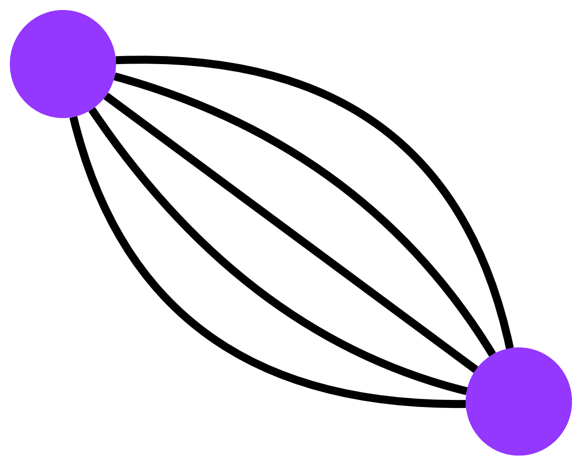


Min-Cut

Cut(G)

```
G′ ← G
while |V(G′)| > 2 do
    e ← uniformly random edge in G′
    G′ ← G′\e
return size of the unique cut in G′
```



5 !

Min-Cut

Cut(G)

For an edge e :

$$Pr[\mu(G) = \mu(G/e)] \geq 1 - \frac{2}{n}$$

For all G with $|V| = n$, $n \geq 3$:

$\hat{p}(G)$:= Probability that Cut(G) returns the value $\mu(G)$

$$\hat{p}(n) := \inf_{\substack{G=(V,E),\\ |V|=n}} \hat{p}(G) \qquad \hat{p}(n) \geq \left(1 - \frac{2}{n}\right) \cdot \hat{p}(n-1) \qquad \hat{p}(n) \geq \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$$

Min-Cut

Cut(G)

We repeat the algorithm Cut(G) $\lambda \binom{n}{2}$ times for some $\lambda > 0$ and return the smallest value obtained.

Runtime : O(λn^4)

Success Probability : The smallest encountered value equals $\mu(G)$ with probability at least $1 - e^{-\lambda}$

$\lambda := lnn$, runtime is O($n^4 logn$) with failure probability $\leq 1/n$

we already had a deterministic solution with this runtime !

Min-Cut

Idea : Last steps are critical

Cut(G) + Strategy Switch in the Critical Region

Stop contracting when there are t vertices remaining

switch to a randomized O(t^4) algorithm with success probability $\geq 1 - e^{-1}$

Runtime : $O(\lambda(\frac{n^4}{t^2} + n^2 t^2) \overset{t=\sqrt{n}}{=} O(\lambda n^3)$

Success Probability : $\geq 1 - e^{-1}$

Bootstrapping : We can use the same method to improve further. In "Limit" we have a O(n^2polylog(n)) algorithm.

Let's take a break

Smallest Enclosing Circle

Smallest Enclosing Circle
Problem Description

given : A finite set of points $P \subseteq \mathbb{R}^2$

to find : The circle with the smallest radius that encloses all points in P

C encloses P :

$C' :=$ the closed disk bounded by C

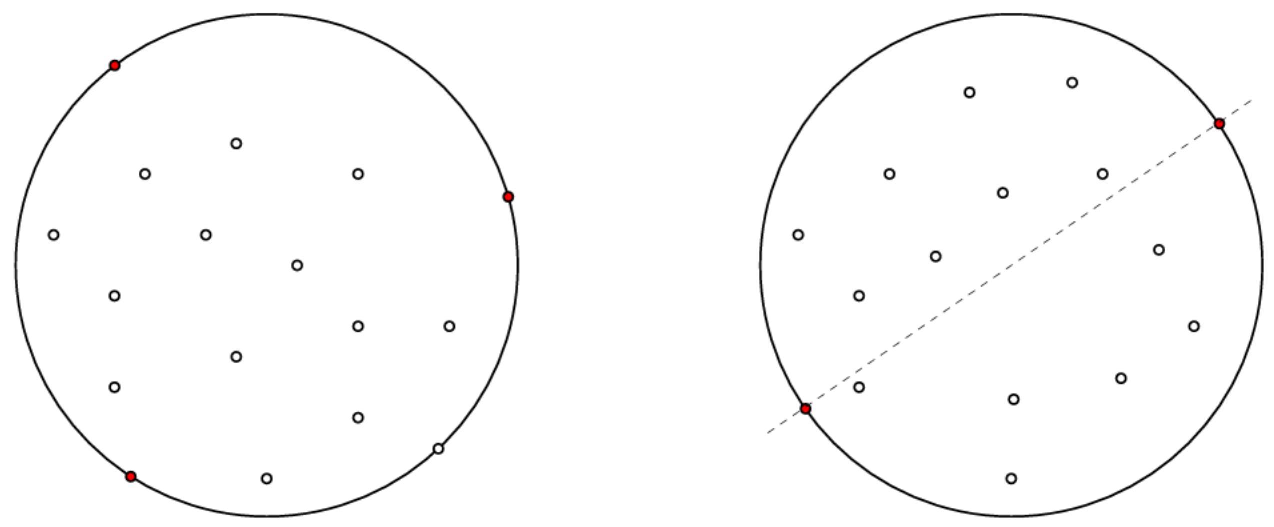
C encloses P if $P \subseteq C'$

Smallest Enclosing Circle

Lemmas

For every finite set of points $P \subseteq \mathbb{R}^2$ there exists a unique smallest enclosing cycle $C(P)$

For every finite set of points $P \subseteq \mathbb{R}^2$ with $|P| \geq 3$ there exists a subset $Q \subseteq P$ with $|Q| = 3$ s.t. $C(Q) = C(P)$



Q acts as a certificate for $C(P)$

Smallest Enclosing Circle

Easy Algorithm

Erster einfacher Algorithmus

$\left|\binom{P}{3}\right| = O(n^3)$ $(|P| = n)$

CompleteEnumeration(P)

```
for all Q ∈ (P choose 3) do
    bestimme C(Q)
    if P ⊆ C•(Q) then
        return C(Q)
```

Für alle Teilmengen $Q \subseteq P$ mit $|Q| = 3$

Falls der kleinste umschliessende Kreis von Q alle Punkte von P enthält, dann ....

Innerer Teil vom Loop: $O(n)$

Gesamte Laufzeit $O(n^4)$

Smallest Enclosing Circle

Algorithm

Runtime : O(n log n)

$P' \leftarrow P$

repeat

    randomly and uniformly choose a subset $Q \subseteq P'$ with $|Q| = 11$

    compute $C(Q)$

    if $P \subseteq C'(Q)$ then return $C(Q)$

    else double all points in P' that lie outside of $C(Q)$

forever

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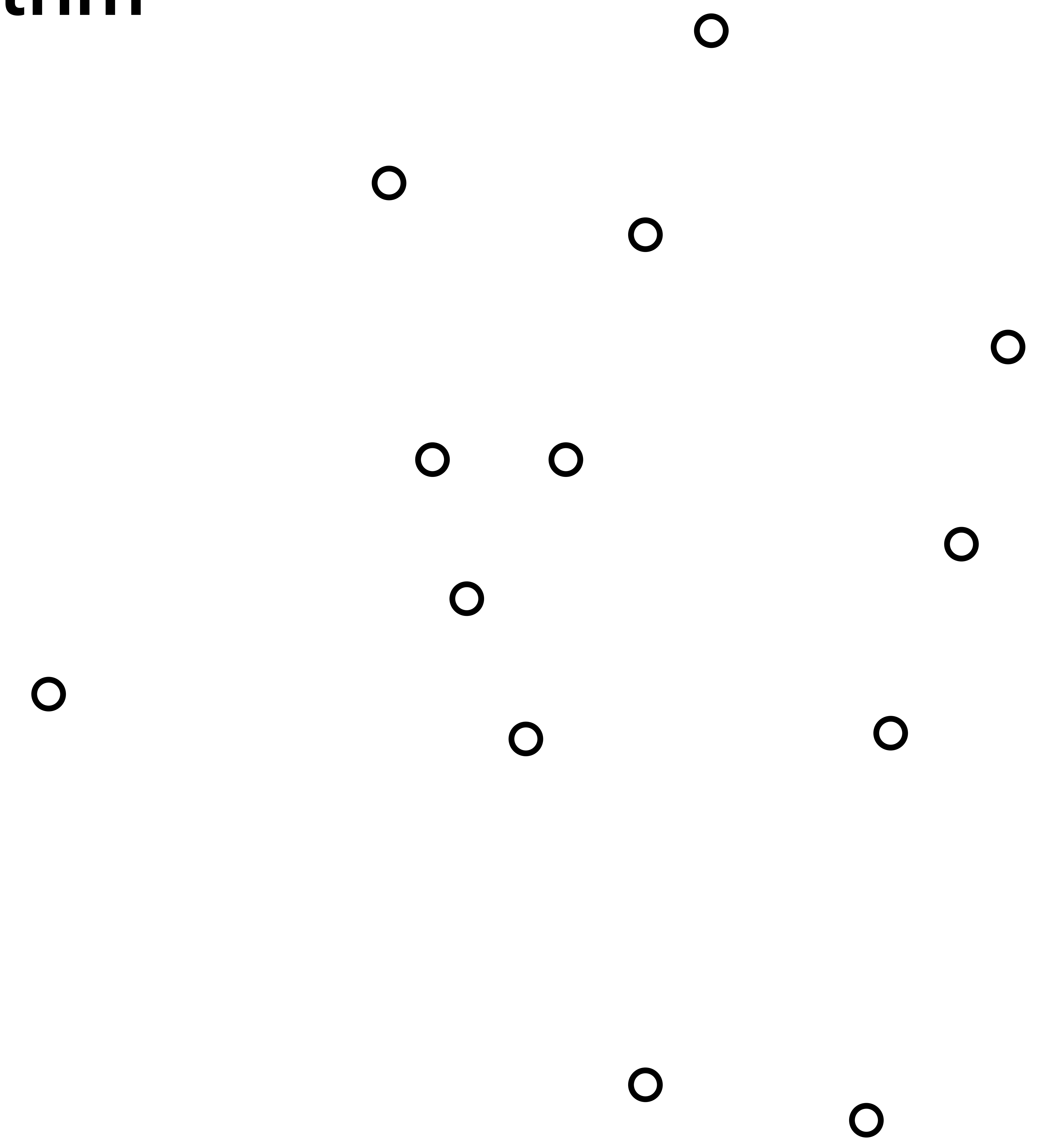

Smallest Enclosing Circle

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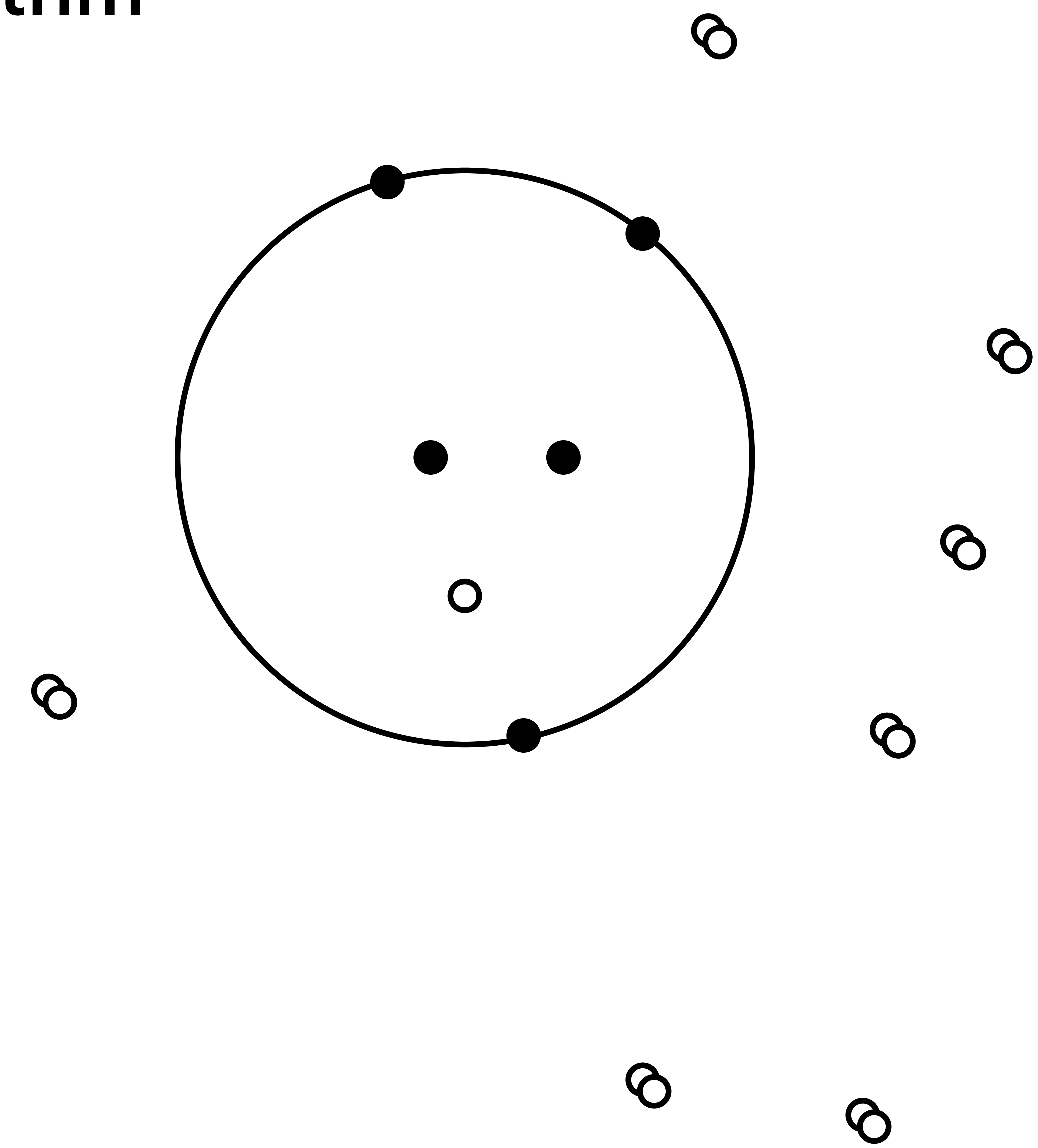

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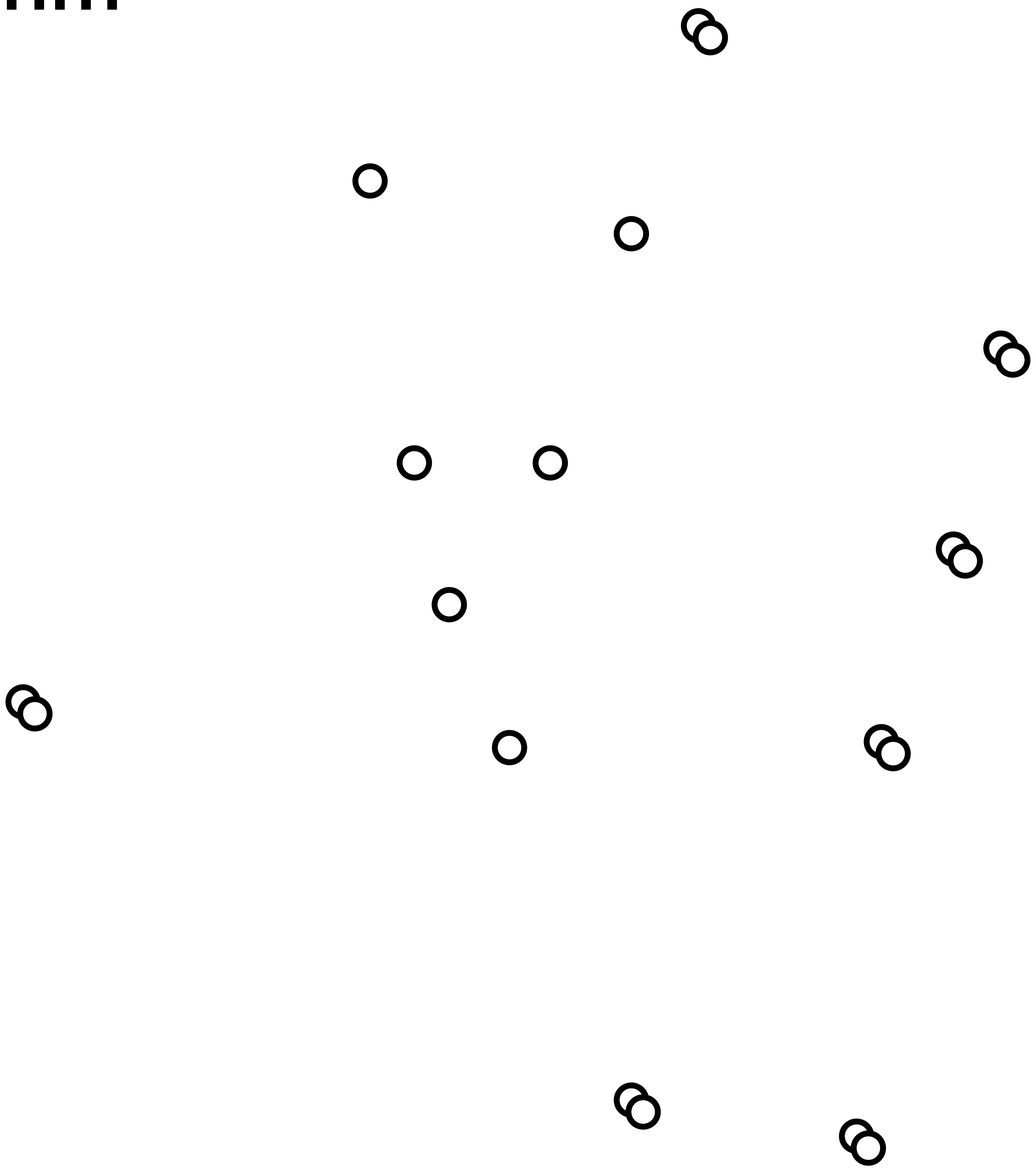

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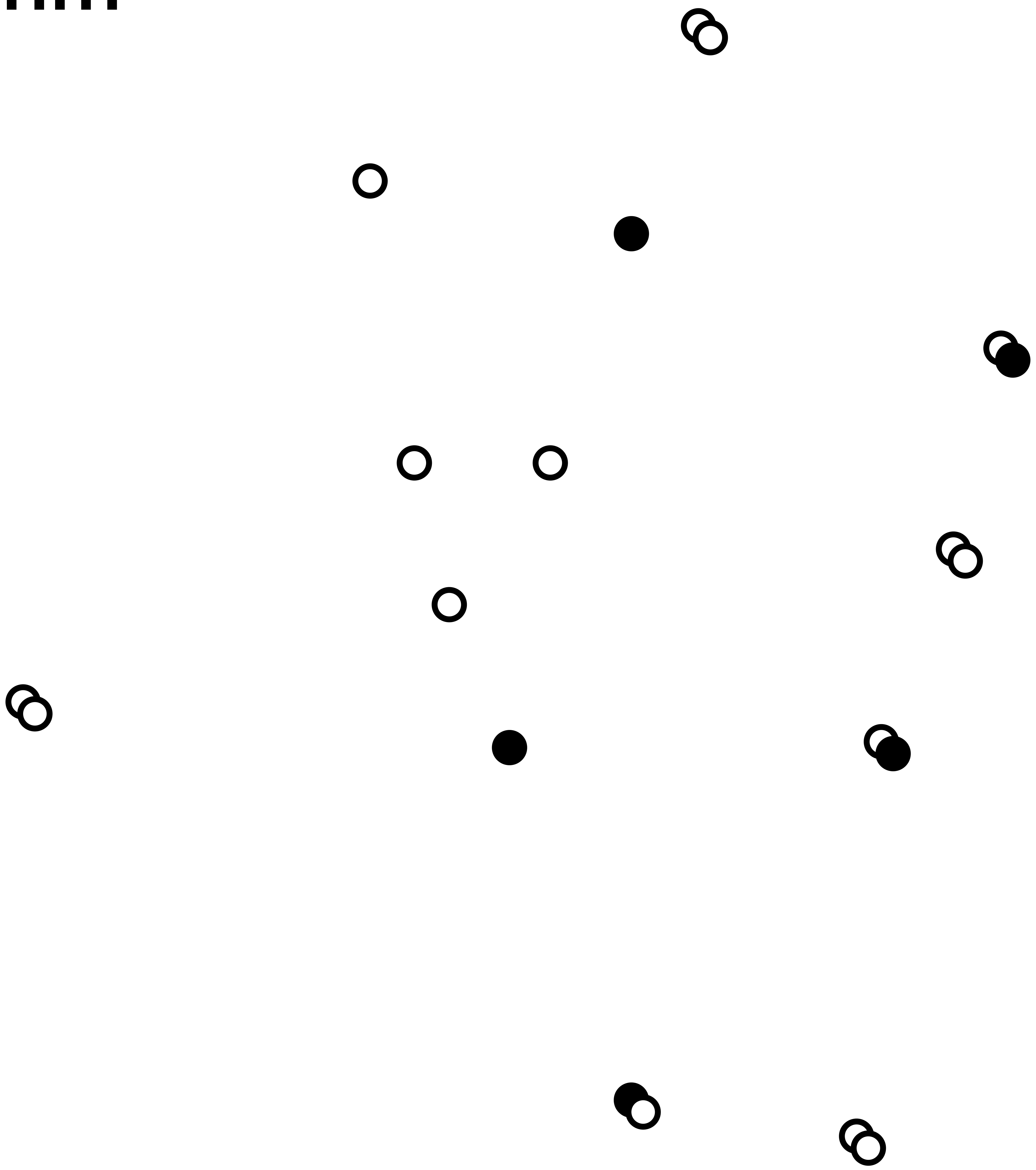

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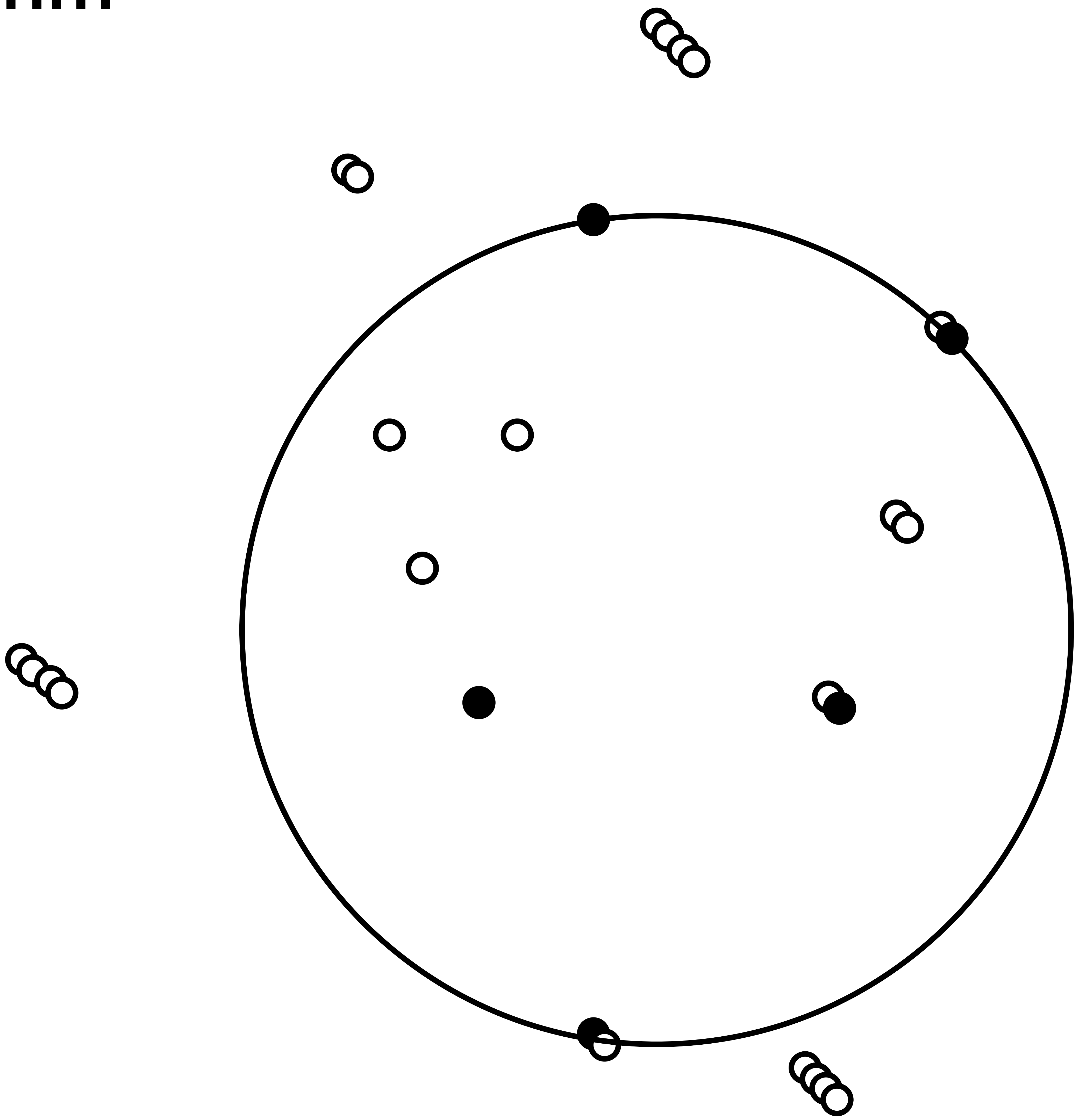

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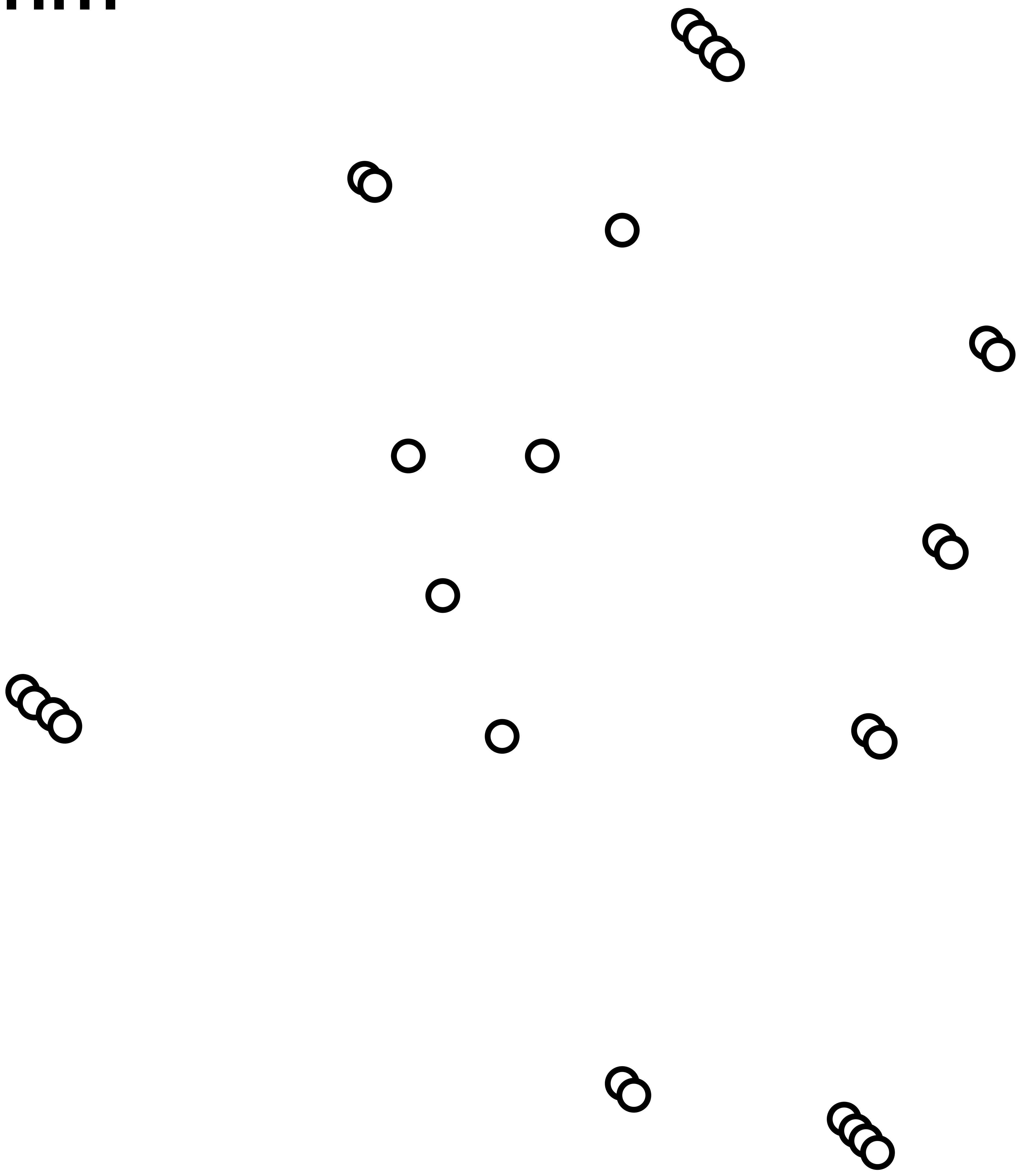

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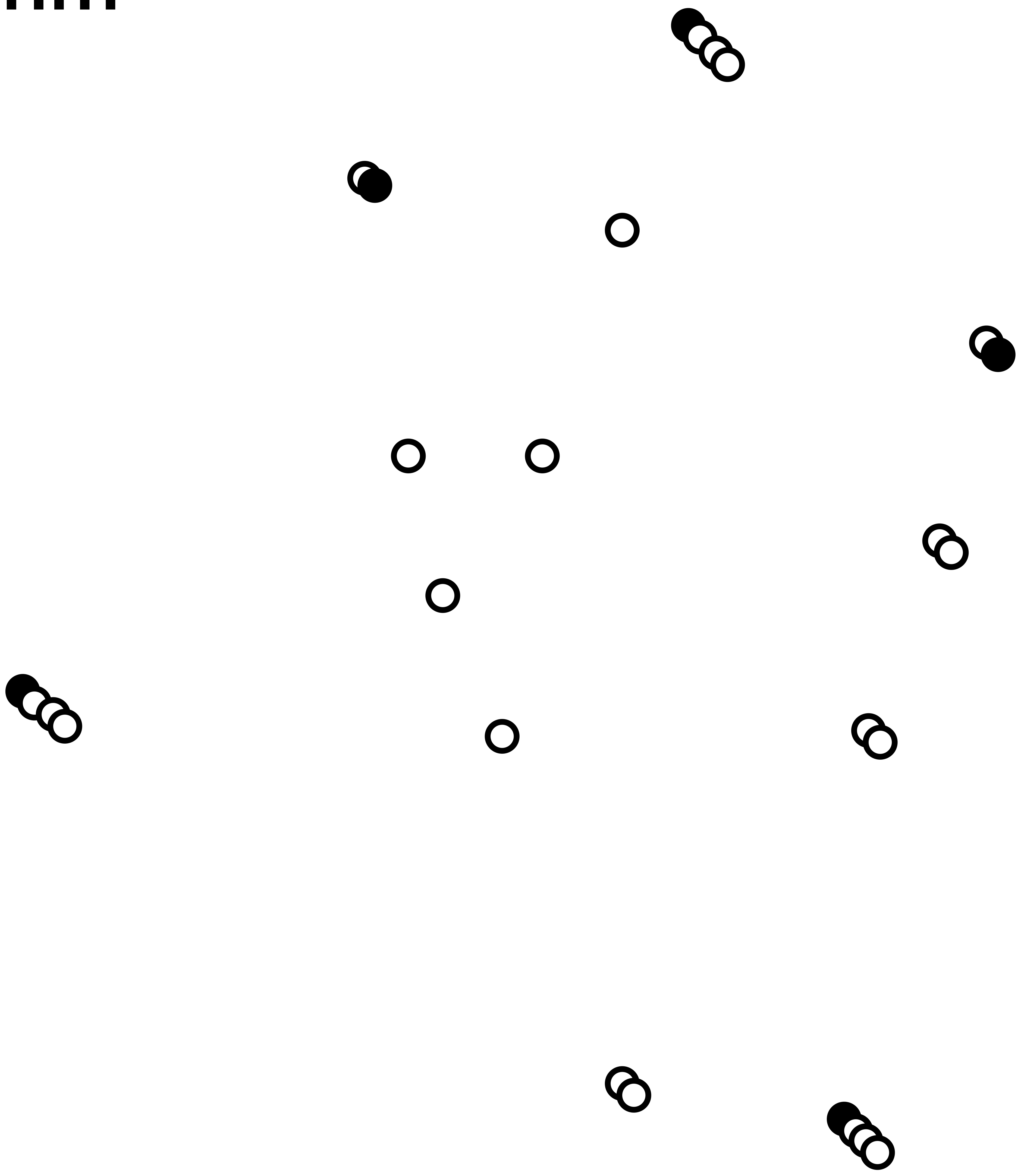

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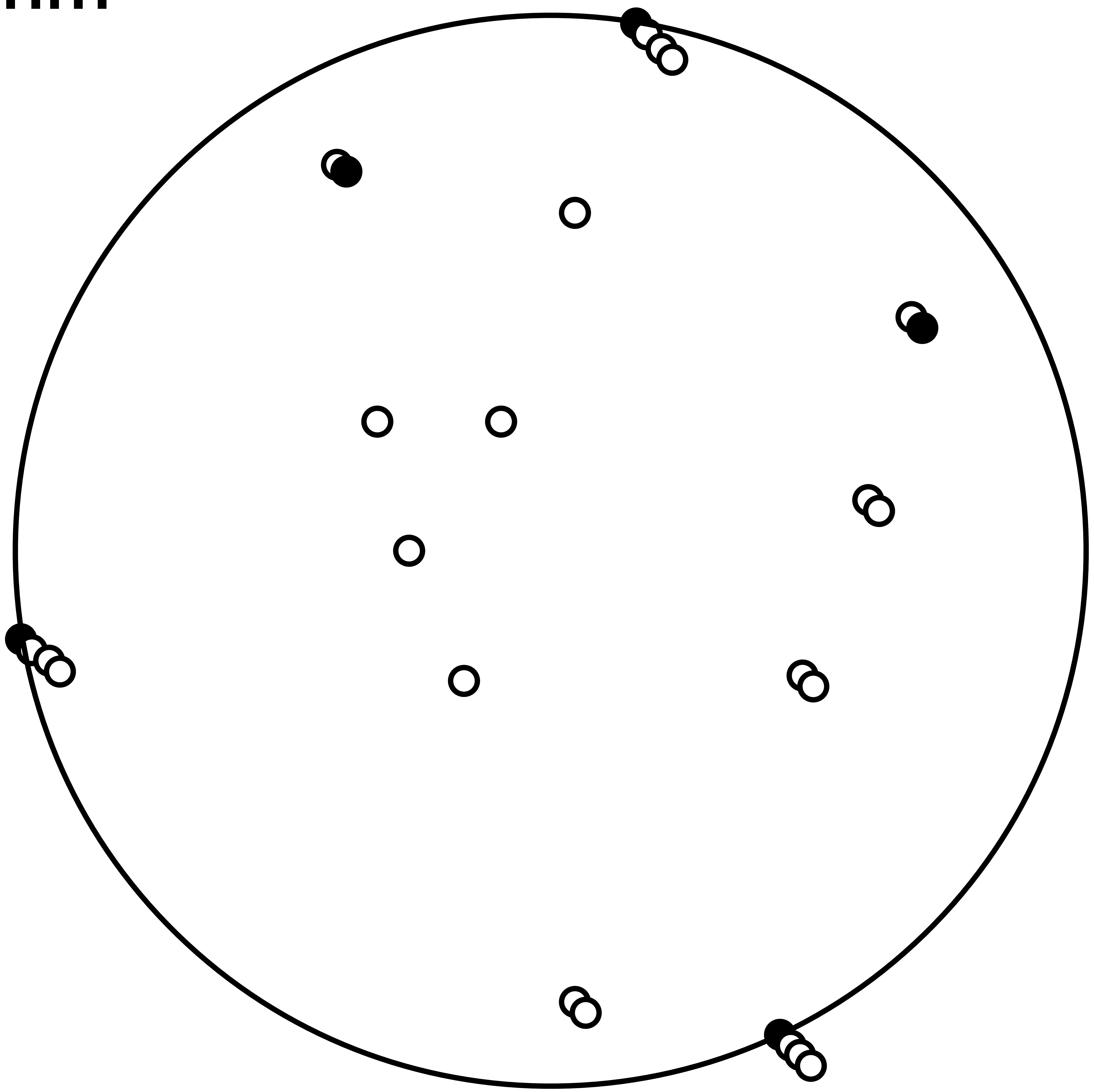

1: $P' \leftarrow P$

2: **repeat**

3: randomly and uniformly choose a subset $Q \subseteq P'$ with $|Q| = 11$

4: compute $C(Q)$

5: if $P \subseteq C'(Q)$ then return $C(Q)$

6: else double all points in P' that lie outside of $C(Q)$

7: **forever**

Smallest Enclosing Circle

Sampling Lemma

Let $r, N \in \mathbb{N}$, $r \leq N$ and $P' \subseteq \mathbb{R}^2$ be a multiset with $|P'| = N$

For R chosen uniformly at random from $\binom{P'}{r}$, the folowing holds :

$$\mathbb{E}\left(\left|\underbrace{P' \backslash C'(R)}_{\text{Points in } P' \text{ outside } C(R)}\right|\right) \leq 3 \cdot \frac{N-r}{r+1} \leq 3 \cdot \frac{N}{r+1}$$

Minitest 5 solutions

Questions

Feedbacks , Recommendations

Nil Ozer