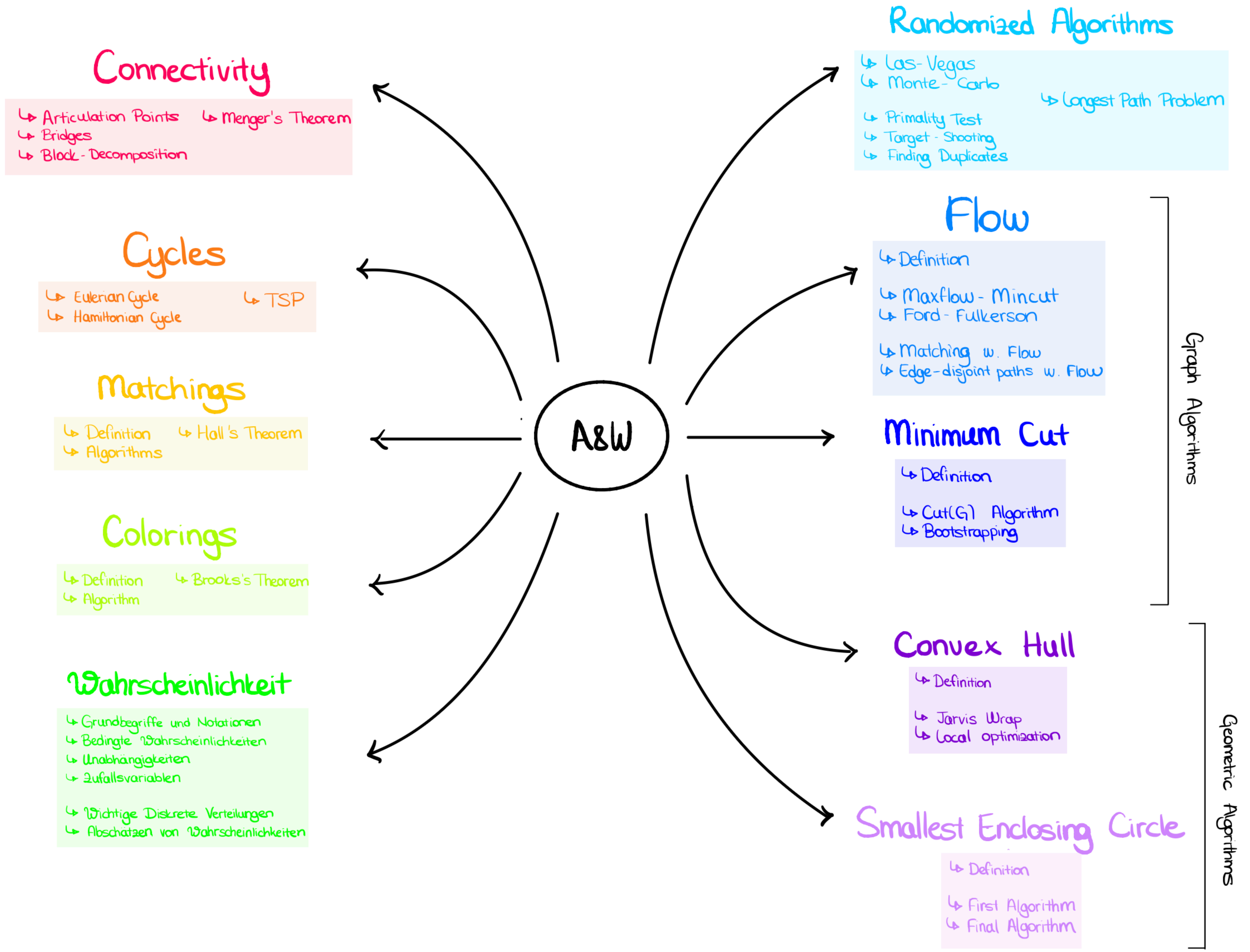
A&W

Exercise Session 10

Randomized Algorithms II

Nil Ozer

A&W Overview


A&W
Connectivity
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Smallest Enclosing Circle
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↳ First Algorithm
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Geometric Algorithms

Last Weeks ...

- 08.05 : Randomized Algorithms II
- 15.05 : Flow
- 22.05 online : Minimum Cut , Convex Hull I (shortly remaining primality tests)
- 27.05/28.05 extra session : Convex Hull II , Smallest Enclosing Cycle
- 29.05 last session : Exam Prep Session + Pizza and Drinks

Outline

- Randomized Algorithms II
  - Randomized Algorithms I recap
  - Primality Tests
  - Colorful Paths

Randomized Algorithms

Recap

Randomized Algorithms

Classic vs. Randomized

$A(I, R)$ can't be reproduced

classic

Input I → Algorithm A → Output $A(I)$

- $A(I)$ is correct and definite for all I
- The runtime is O(f(n)) for all I with $|I| = n$

randomized

Input I → Algorithm A → Output $A(I, R)$

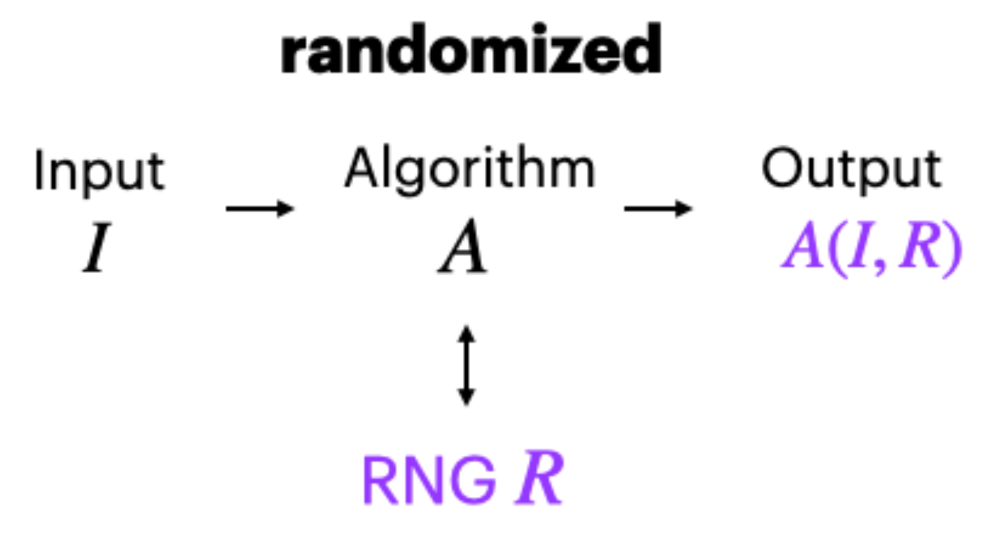
↕

RNG R

- $A(I, R)$ is correct with $Pr_R[A(I, R) \ is \ correct] \geq$ ... for all I
- The runtime is O(f(n)) and/or Pr_R[Runtime $\leq$ O(f(n))] $\geq$ ... for all I with $|I| = n$

Randomized Algorithms

Las-Vegas vs. Monte-Carlo



Las-Vegas

Runtime is the RV

- can output true answer
- cannot output false answer
- can run forever/ can output no answer (???)

Monte-Carlo

Correctness/Quality is the RV

- can output true answer
- can output false answer
- always outputs an answer

Input: An array of $n \geq 2$ elements, in which half are '*a*'s and the other half are '*b*'s.

Output: Find an '*a*' in the array.

```
findingA_LV(array A, n)
begin
    repeat
        Randomly select one element out of n elements.
    until 'a' is found
end
```

```
findingA_MC(array A, n, k)
begin
    i := 0
    repeat
        Randomly select one element out of n elements.
        i := i + 1
    until i = k or 'a' is found
end
```

Target-Shooting

Problem Description

given : finite sets S and U with $S \subseteq U$

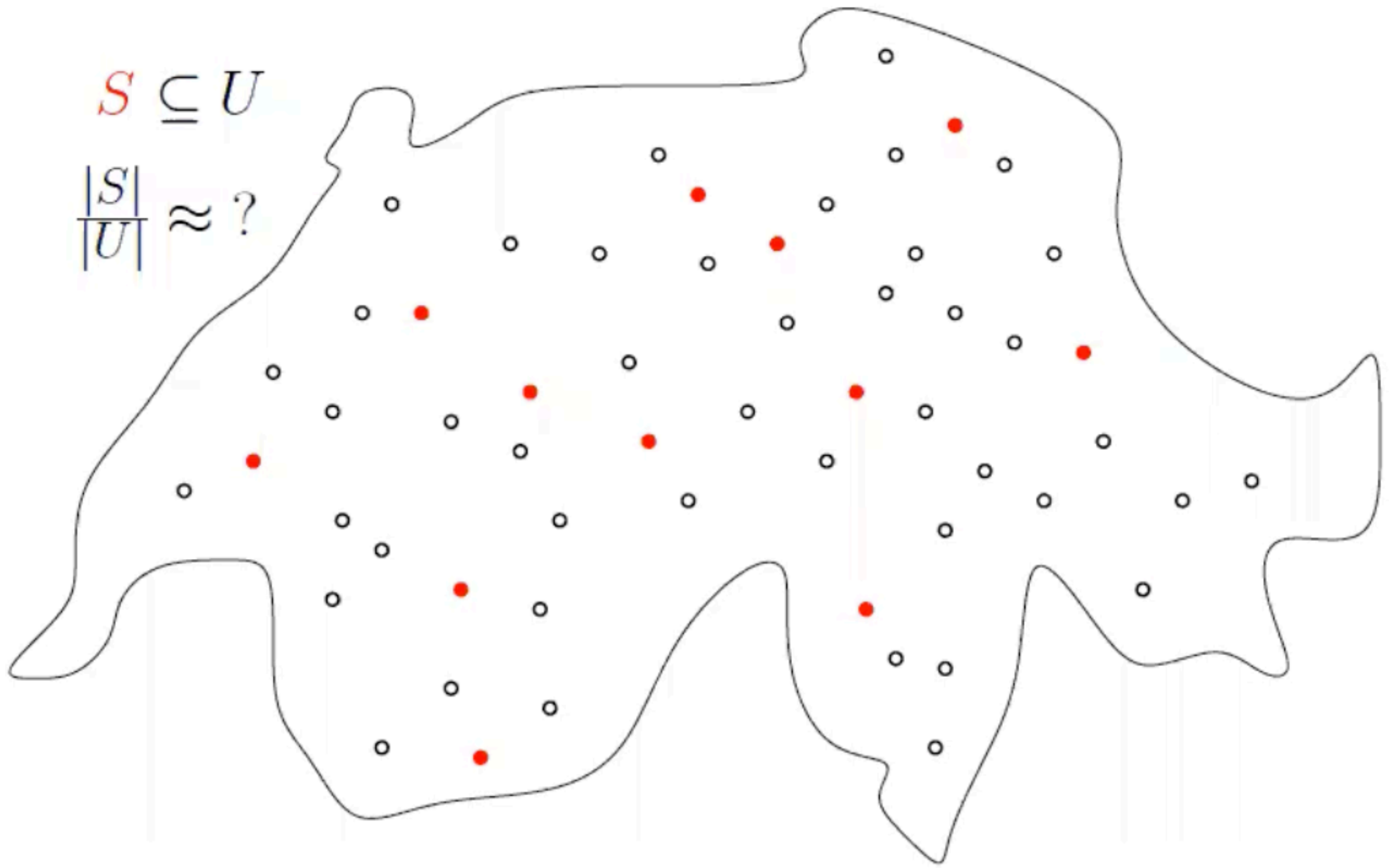
to find : $\approx \frac{|S|}{|U|}$

U is very large. We cannot afford to iterate through U

We can generate elements u in U uniformly distributed

$I_S : U \rightarrow \{0,1\}$

$I_S(u) = 1 \iff u \in S$



Target-Shooting
Algorithm

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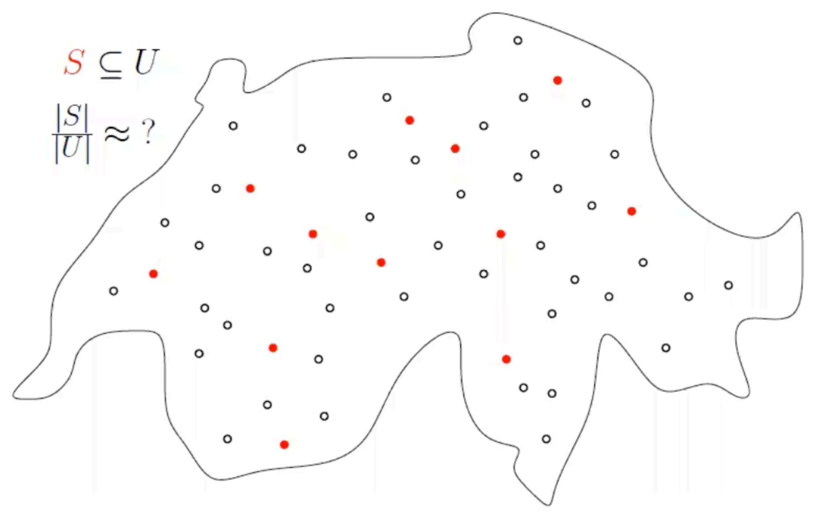

1 : Pick $u_1, \ldots, u_N$ from U randomly, uniformly and independently

2 : Return $\frac{1}{N} \cdot \sum_{i=1}^{N} I_S(u_i)$

Target-Shooting
Algorithm

given : finite sets S and U with $S \subseteq U$

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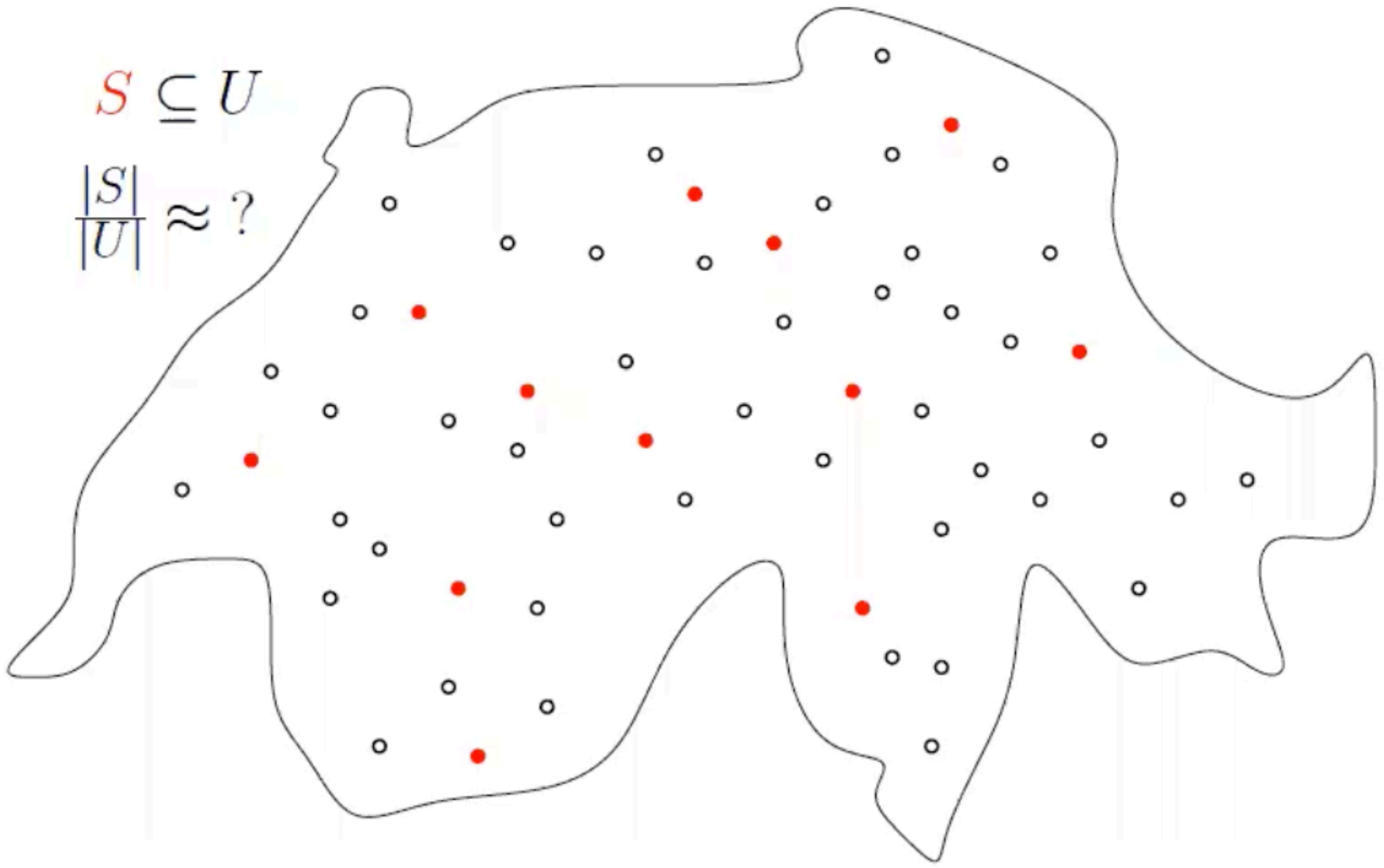

Target-Shooting
Algorithm

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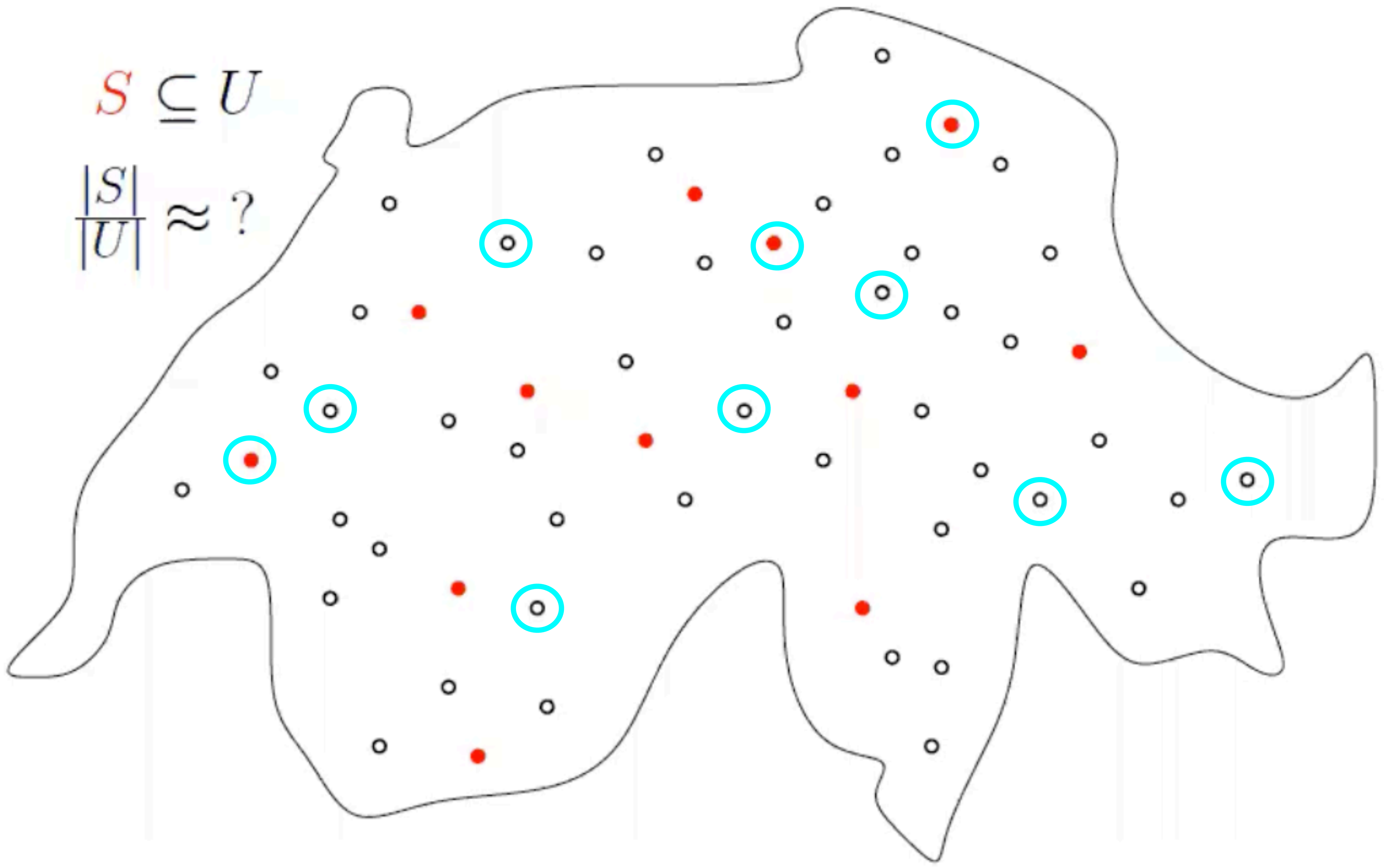


Target-Shooting
Algorithm

given : finite sets S and U with $S \subseteq U$

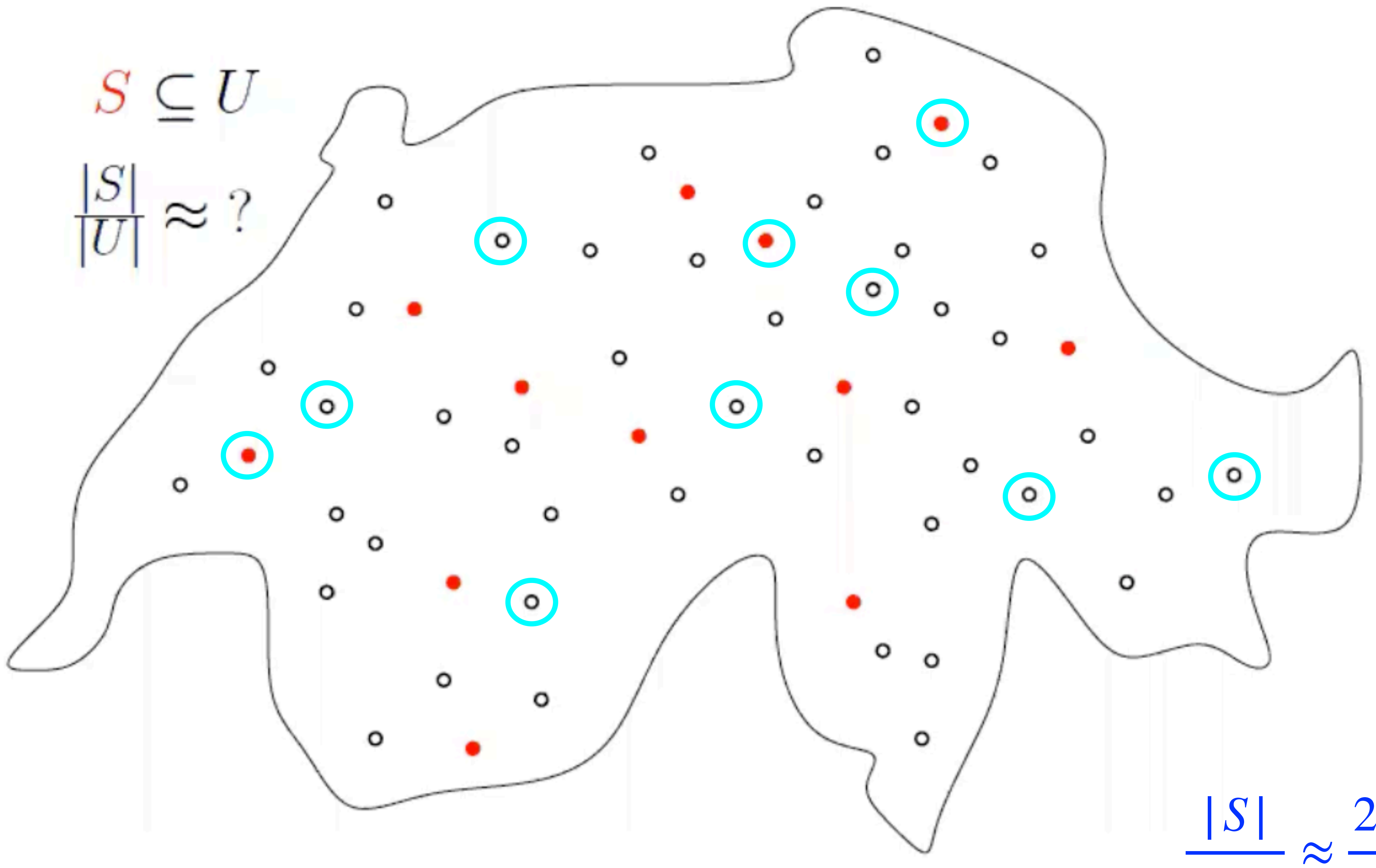
to find : $\approx \frac{|S|}{|U|}$

$$I_S(u) = 1 \iff u \in S$$

1 : Pick $u_1, \ldots, u_N$ from U randomly, uniformly and independently

2 : Return $\frac{1}{N} \cdot \sum_{i=1}^{N} I_S(u_i)$

$$\frac{1}{10} \cdot \sum_{i=1}^{10} I_S(u_i) = \frac{3}{10}$$



$$\frac{|S|}{|U|} \approx \frac{20}{64} = 0.3125$$

Target-Shooting
Algorithm

given : finite sets S and U with $S \subseteq U$

to find : $\approx \frac{|S|}{|U|}$

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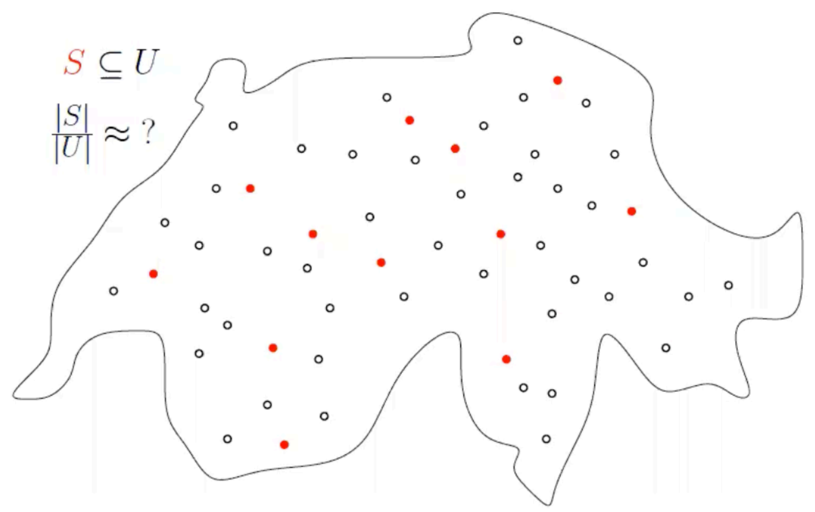

1 : Pick $u_1, \ldots, u_N$ from U randomly, uniformly and independently

2 : Return $\frac{1}{N} \cdot \sum_{i=1}^{N} I_S(u_i)$

Fehlerreduktionen:

- **Wiederholung MC:** Eine N-fache Wiederholung mit $N = 4\varepsilon^{-2} \ln \delta^{-1}$ steigert die Erfolgswahrscheinlichkeit eines Monte-Carlo-Algorithmus von $\frac{1}{2} + \varepsilon$ auf $\geq 1 - \delta$.
- **Wiederholung MC mit einseitigem Fehler:** Eine N-fache Wiederholung mit $N = \varepsilon^{-1} \ln \delta^{-1}$ steigert fÃ¼r einen Monte-Carlo-Algorithmus mit einseitigem Fehler die Erfolgswahrscheinlichkeit von ε auf $\geq 1 - \delta$.
- **Target Shooting:** Bestimmt der Target-Shooting-Algorithmus eine Menge $S \subseteq U$ mit $N \geq 3\frac{|U|}{|S|}\varepsilon^{-2} \ln(2/\delta)$ Versuchen, so ist die Ausgabe mit Wahrscheinlichkeit $\geq 1 - \delta$ im Intervall $\left[(1-\varepsilon)\frac{|S|}{|U|}, (1+\varepsilon)\frac{|S|}{|U|}\right]$.

Finding Duplicates

Problem Description

given : A dataset $D = (s_1, s_2, \ldots, s_n)$, is a sequence of n elements

to find : find all duplicates in D

(i, j) with $1 \leq i < j \leq n$ is a duplicate in D if $s_i = s_j$

$$\mathcal{D} = (\underset{1}{A}, \underset{2}{C}, \underset{3}{B}, \underset{4}{Z}, \underset{5}{C}, \underset{6}{B}, \underset{7}{C})$$

$$\text{Dupl}(\mathcal{D}) = \{(2,5), (2,7), (5,7), (3,6)\}$$

Finding Duplicates
Problem Description

Elements in D are very large.

Storing and comparing is expensive

given : A dataset $D = (s_1, s_2, \ldots, s_n)$, is a sequence of n elements

to find : find all duplicates in D (i, j) with $1 \leq i < j \leq n$ is a duplicate in D if $s_i = s_j$

Hashfunction h :

$h : U \to [m]$ $[m] = \{1,2,\ldots,m\}$

h is efficiently computable
h behaves like a random variable

$\forall u \in U \ \forall i \in [m] :$ $\Pr[h(u) = i] = \frac{1}{m}$ (independent for different u)

Finding Duplicates
Problem Description

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Each $h(s_i)$ is uniformly randomly distributed in $[m]$ BUT

$$s_i = s_j \implies h(s_i) = h(s_j)$$

Our m is much smaller than $|U|$ (compression)

Finding Duplicates

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	A	C	B	Z	C	B	C
hashing:	$(\underbrace{\mathbf{31}}_{h(A)}, 1)$	$(\underbrace{\mathbf{27}}_{h(C)}, 2)$	$(\underbrace{\mathbf{12}}_{h(B)}, 3)$	$(\underbrace{\mathbf{12}}_{h(Z)}, 4)$	$(\underbrace{\mathbf{27}}_{h(C)}, 5)$	$(\underbrace{\mathbf{12}}_{h(B)}, 6)$	$(\underbrace{\mathbf{27}}_{h(C)}, 7)$
sorting:	$(\mathbf{12}, 3)$	$(\mathbf{12}, 4)$	$(\mathbf{12}, 6)$	$(\mathbf{27}, 2)$	$(\mathbf{27}, 5)$	$(\mathbf{27}, 7)$	$(\mathbf{31}, 1)$
duplicates:		$(3, 6)$,			$(2, 5), (2, 7), (5, 7)$		

Finding Duplicates

Challenge : Collisions

given : A dataset $D = (s_1, s_2, \ldots, s_n)$, is a sequence of n elements

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$$A \quad C \quad B \quad Z \quad C \quad B \quad C$$

hashing: $(\underbrace{\mathbf{31}}_{h(A)}, 1)$ $(\underbrace{\mathbf{27}}_{h(C)}, 2)$ $(\underbrace{\mathbf{12}}_{h(B)}, 3)$ $(\underbrace{\mathbf{12}}_{h(Z)}, 4)$ $(\underbrace{\mathbf{27}}_{h(C)}, 5)$ $(\underbrace{\mathbf{12}}_{h(B)}, 6)$ $(\underbrace{\mathbf{27}}_{h(C)}, 7)$

sorting: $(\mathbf{12}, 3)$ $(\mathbf{12}, 4)$ $(\mathbf{12}, 6)$ $(\mathbf{27}, 2)$ $(\mathbf{27}, 5)$ $(\mathbf{27}, 7)$ $(\mathbf{31}, 1)$

duplicates: $(3, 6)$, $(2, 5), (2, 7), (5, 7)$

collision : $h(B) = h(Z)$

Finding Duplicates

Challenge : Collisions

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Collision :

The new, undesired duplicates in the hashmap

the pairs (i, j) , $1 \leq i < j \leq n,$ with $s_i \neq s_j$ and $h(s_i) = h(s_j)$

Finding Duplicates

Challenge : Collisions

given : A dataset $D = (s_1, s_2, \ldots, s_n)$, is a sequence of n elements

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$\mathbb{E}$[#Collisions] :

$K_{i,j}$ bernoulli RV. with : $K_{i,j} = 1 \iff (i,j)$ is a collision

$$\Pr[K_{i,j} = 1] = \begin{cases} \frac{1}{m} & \text{if } s_i \neq s_j \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{E}[K_{i,j}] \le \frac{1}{m}$$

$$\mathbb{E}[\#\text{Collisions}] = \sum_{1 \le i < j \le n} \mathbb{E}[K_{i,j}] \le \binom{n}{2} \cdot \frac{1}{m}$$

given : A dataset $D = (s_1, s_2, ..., s_n)$, is a sequence of n elements

to find : find all duplicates in D (i, j) with $1 \leq i < j \leq n$ is a duplicate in D if $s_i = s_j$

Hashfunction h :

$h : U \rightarrow [m]$ $[m] = \{1,2,...,m\}$ $\forall u \in U \ \forall i \in [m] :$ $\Pr[h(u) = i] = \frac{1}{m}$

Each $h(s_i)$ is uniformly randomly distributed in $[m]$ BUT

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Finding Duplicates

Challenge : Collisions

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$$\Pr[K_{i,j} = 1] = \begin{cases} \frac{1}{m} & \text{if } s_i \neq s_j \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{E}[K_{i,j}] \leq \frac{1}{m}$$

$$\mathbb{E}[\#\text{Collisions}] \leq \binom{n}{2} \cdot \frac{1}{m} < 1 \quad \text{for } m = n^2$$

Finding Duplicates

Runtime

given : A dataset $D = (s_1, s_2, \ldots, s_n)$, is a sequence of n elements

to find : find all duplicates in D (i,j) with $1 \le i < j \le n$ is a duplicate in D if $s_i = s_j$

Hashfunction h :

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Collision :

The new, undesired duplicates in the hashmap

the pairs (i,j) , $1 \le i < j \le n$, with $s_i \neq s_j$ and $h(s_i) = h(s_j)$

$$\mathbb{E}[\#\text{Collisions}] \le \binom{n}{2} \cdot \frac{1}{m} < 1 \quad \text{for } m = n^2$$

Runtime :

- n hash computations
- sorting in O(n log n)
- duplicate check comparisons (|Dupl(D)|+#Kollisionen) ≈ O(n)

additional memory

$$\underbrace{O(n \log n)}_{\text{indices}} + \underbrace{O(n \log m)}_{\text{hash values}} \overset{m=n^2}{=} O(n \log n)$$

Overall : O(n log n)

Randomized Algorithms

Primality Tests

Primality Test

Problem Description

given : A number $n \in \mathbb{N}$

to find : is n prime ? ?

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	...
$f(n)$	0	1	0	1	2	1	0	4	2	1	8	1	2	4	0	1	14	1	8	4	2	1	...

Primality Test

Problem Description

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	...
$f(n)$	0	1	0	1	2	1	0	4	2	1	8	1	2	4	0	1	14	1	8	4	2	1	...

given : A number $n \in \mathbb{N}$

to find : is n prime $\iff$ n has no divider in $\{2, \ldots, n-1\}$

prime-counting function $\pi(x)$:

$$\pi(x) := \left| \{ n \in \mathbb{N} \mid n \leq x, n \text{ prime} \} \right| \sim \frac{x}{\ln x}$$

Primality Test
Problem Description

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	...
$f(n)$	0	1	0	1	2	1	0	4	2	1	8	1	2	4	0	1	14	1	8	4	2	1	...

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2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, ...

$$\pi(11) =$$

Primality Test

Problem Description

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	...
$f(n)$	0	1	0	1	2	1	0	4	2	1	8	1	2	4	0	1	14	1	8	4	2	1	...

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$$\pi(x) := \left| \{ n \in \mathbb{N} \mid n \leq x, n \text{ prime} \} \right| \sim \frac{x}{\ln x}$$

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, ...

$$\pi(11) = 5$$

Primality Test
Problem Description

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	...
$f(n)$	0	1	0	1	2	1	0	4	2	1	8	1	2	4	0	1	14	1	8	4	2	1	...

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prime-counting function $\pi(x)$: $\pi(x) := \left| \{ n \in \mathbb{N} \mid n \leq x, n \text{ prime} \} \right| \sim \frac{x}{\ln x}$

Primality Test
Naive Algorithm

given : A number $n \in \mathbb{N}$

to find : is n prime $\iff$ n has no divider in $\{2, \ldots, n-1\}$

prime-counting function $\pi(x)$: $\pi(x) := \left|\{n \in \mathbb{N} \mid n \leq x, n \text{ prime}\}\right| \sim \frac{x}{\ln x}$

1) For all $a \leq \sqrt{n}$ test if a divides n

Primality Test

Easy randomized test

given : A number $n \in \mathbb{N}$

to find : is n prime $\iff$ n has no divider in $\{2, \dots, n-1\}$

prime-counting function $\pi(x)$: $\pi(x) := \left| \{ n \in \mathbb{N} \mid n \leq x, n \text{ prime} \} \right| \sim \frac{x}{\ln x}$

1) Choose $a \in \{1, 2, \dots, \sqrt{n}\}$ uniformly at random

2) if a divides n then return 'not prime'

3) else return 'prime'

Refresher

DiskMat 🤢

gcd : greatest common divisor

$$n \text{ is prime} \Rightarrow \gcd(a, n) = 1 \quad \forall a \in [1, n-1]$$

$\mathbb{Z}_n^*$: the multiplicative group modulo n

$$\mathbb{Z}_n^* = \left\{ a \in \{1, 2, \ldots, n-1\} \mid \gcd(a, n) = 1 \right\}$$

Primality Test

Euclidean Primality Test

given : A number $n \in \mathbb{N}$

to find : is n prime $\iff$ n has no divider in $\{2, \ldots, n-1\}$

prime-counting function $\pi(x)$: $\pi(x) := \left|\{n \in \mathbb{N} \mid n \leq x, n \text{ prime}\}\right| \sim \frac{x}{\ln x}$

n is prime $\Rightarrow \gcd(a, n) = 1 \quad \forall a \in [1, n-1]$

gcd := greatest common divisor

can be calculated in O((log nm)³)

1) Choose $a \in \{1, 2, \ldots, \sqrt{n}\}$ uniformly at random

2) if $gcd(a, n) > 1$ then return 'not prime'

3) else return 'prime'

- if n is a prime : always correct
- if n is not a prime : it might return a wrong answer with the probability

$$\frac{\left|\{a \in [1, n-1] : \gcd(a, n) = 1\}\right|}{n-1} = \frac{|\mathbb{Z}_n^*|}{n-1}$$

Let's take a break

A&W 😊🧑🏻‍💻

WhatsApp group

Randomized Algorithms

Colorful Paths

Helper

Mathematical Tools and Notations

$[n] := \{1,2,\ldots,n\}$

$[n]^k :=$ the set of sequences over $[n]$ of length k $\qquad |[n]^k| = n^k$

$\binom{[n]}{k} :=$ the set of k-element subsets of $[n]$ $\qquad \left|\binom{[n]}{k}\right| = \binom{n}{k}.$

The k nodes on a path of length $k-1$ can be colored using $[k]$ in exactly k^k ways

$k!$ of these colorings use each color exactly once

Helper

Mathematical Tools and Notations

Handshaking lemma : For all graphs , it holds that $\sum_{v \in V} \deg(v) = 2|E|$.

If you repeat an experiment with success probability p until success, then the expected number of trials is $\frac{1}{p}$ ($Geo(p)$)

Helper

Mathematical Tools and Notations

For $c, n \in \mathbb{R}^+$, it holds that $c^{\log n} = n^{\log c}$

$$2^{\log n} = n^{\log 2} = n \text{ and } 2^{\mathcal{O}(\log n)} = n^{\mathcal{O}(1)} \text{ is always polynomial in } n$$

For $n \in \mathbb{N}_0$, it holds that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ (binomial theorem)

For $n \in \mathbb{N}_0$, it holds that $\frac{n!}{n^n} \geq e^{-n}$ (power series expansion of the exponential function)

Long-Path

Problem Description

given : A graph G and a number $B \in \mathbb{N}_0$

to find : is there a path of length B in G

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NP-Complete

Detour !

Colorful Paths
Problem Description

given : A graph $G = (V, E)$

A coloring of its vertices with k colors $\gamma : V \to [k]$

to find : Does there exist a colorful path of length $k - 1$ in a randomly colored graph ?

colorful :

A path is colorful if all of the vertices in the path have a different color

Colorful Paths

Problem Description

given : A graph $G = (V, E)$

A coloring of its vertices with k colors $\gamma : V \to [k]$

to find : Does there exist a colorful path of length $k - 1$ in a randomly colored graph ?

A path is colorful if all of the vertices in the path have a different color

$P_i(v) := \{S \subseteq [k], |S| = i + 1 \ \mid$ There exists a colorful path of length i ending in v with colors $S\}$

$\exists$ colorful path of length $k - 1 \iff \bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$

Colorful Paths

$P_i(v)$

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color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

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- $P_0(v) =$

Colorful Paths

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Colorful Paths

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- $P_1(v) = \{\{\gamma(x), \gamma(v)\} \mid x \in N(v) \text{ und } \gamma(x) \neq \gamma(v)\}$

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Colorful Paths

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- $P_i(v) = \bigcup_{x \in N(v)} \{R \cup \{\gamma(v)\} \mid R \in P_{i-1}(x) \text{ und } \gamma(v) \notin R\}$

Colorful Paths
Algorithm

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Algorithm 1: COLORFUL(G, i) — G a γ-colored graph

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i-1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

Algorithm 2: RAINBOW(G, γ) — G a graph, γ a k-coloring

```
forall v ∈ V do
    P_0(v) ← {{γ(v)}};
for i = 1 to k − 1 do
    COLORFUL(G, i);
return ⋃_{v∈V} P_{k−1}(v) ≠ ∅;
```

Colorful Paths
Algorithm

Algorithm 2: RAINBOW(G, γ)

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
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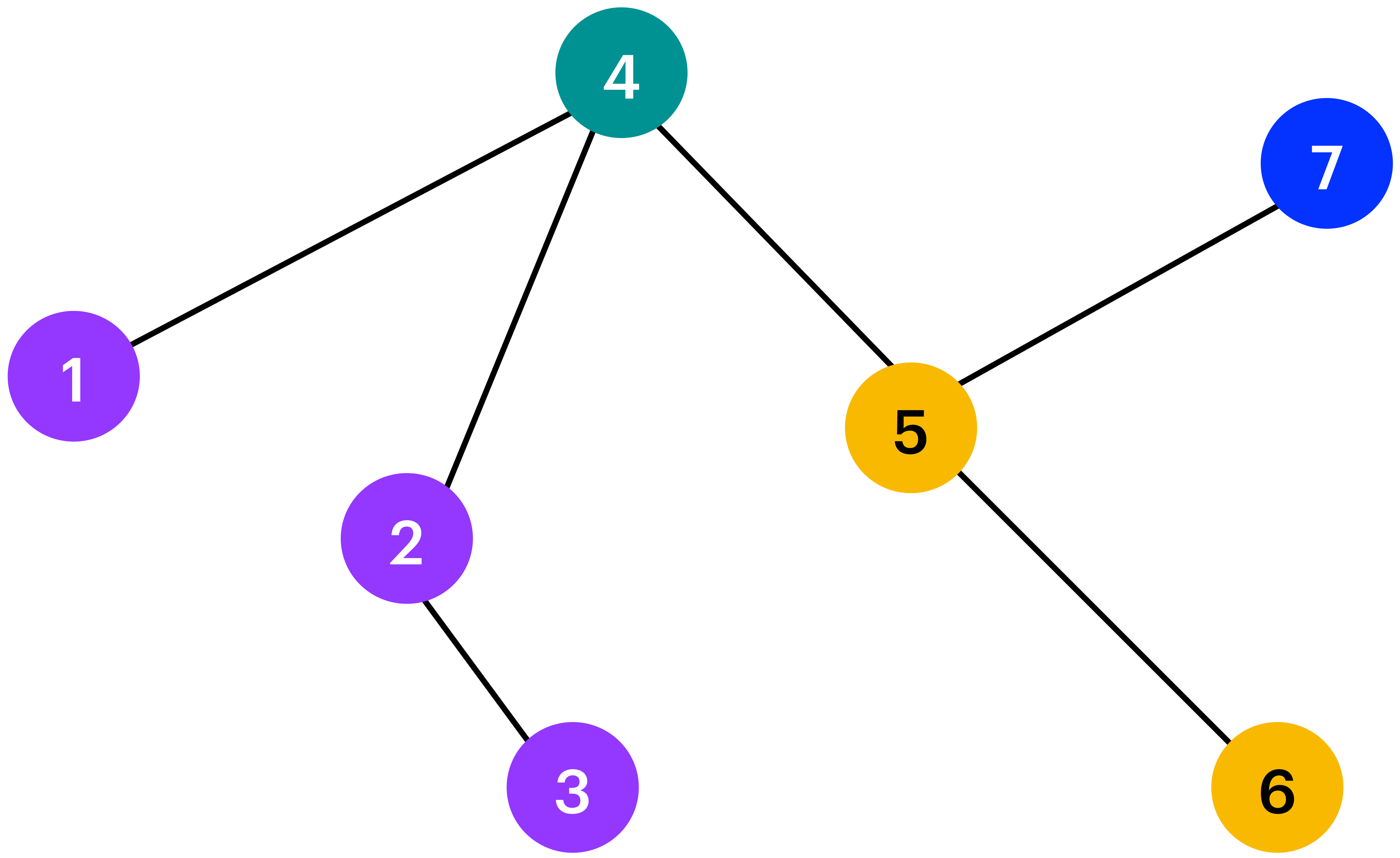
P_0	
P_0 (1)	
P_0 (2)	
P_0 (3)	
P_0 (4)	
P_0 (5)	
P_0 (6)	
P_0 (7)	



γ 1 , 2 , 3 , 4

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

Colorful Paths

Algorithm

Algorithm 2: $\textsc{Rainbow}(G, \gamma)$

forall $v \in V$ **do**
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for $i = 1$ **to** $k - 1$ **do**
  $\textsc{Colorful}(G, i)$;
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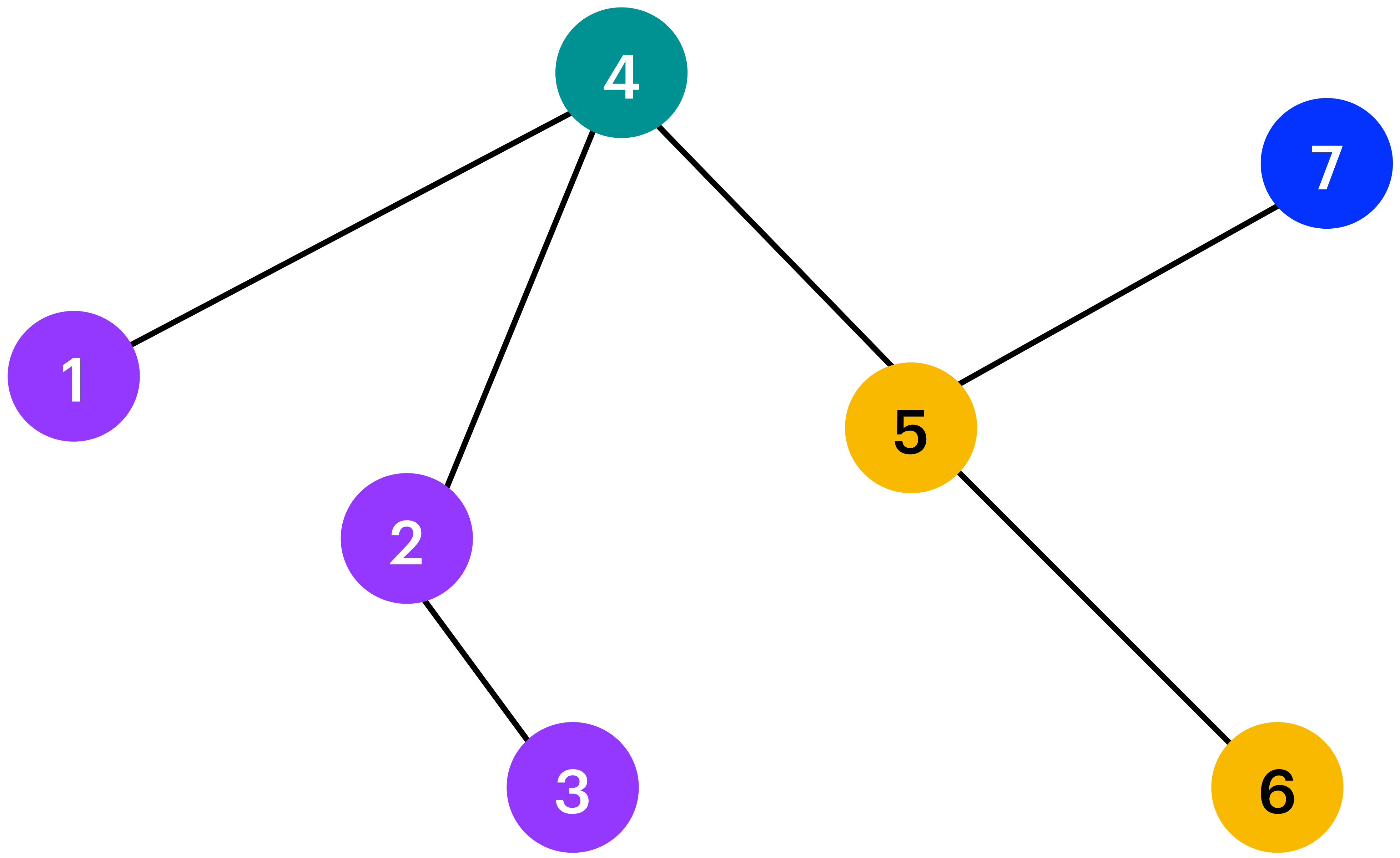
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}



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Colorful Paths

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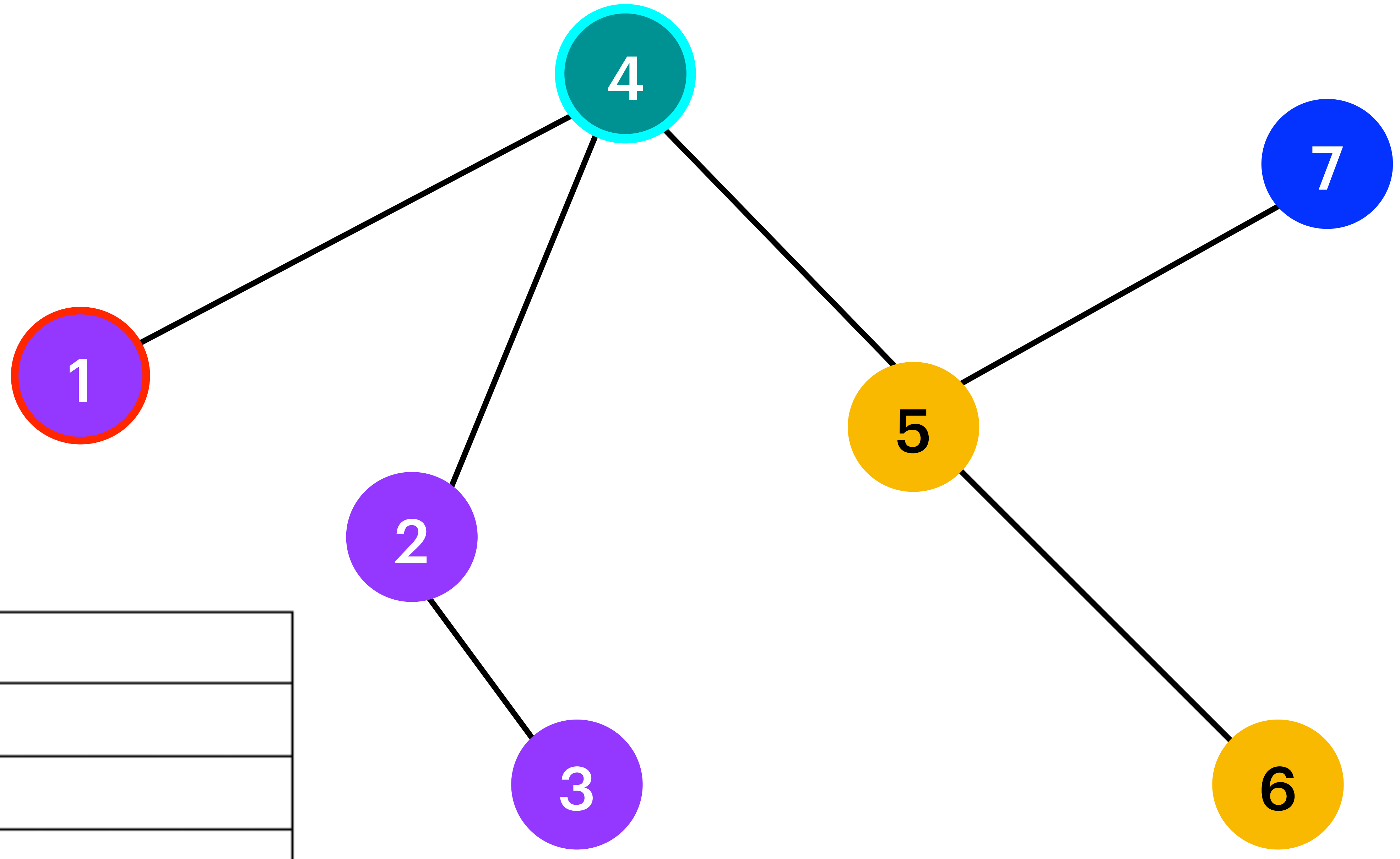
P_1	
$P_1(1)$	
$P_1(2)$	
$P_1(3)$	
$P_1(4)$	
$P_1(5)$	
$P_1(6)$	
$P_1(7)$	

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}



γ 1 , 2 , 3 , 4

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P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	
$P_1(3)$	
$P_1(4)$	
$P_1(5)$	
$P_1(6)$	
$P_1(7)$	

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P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
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$P_0(5)$	{{3}
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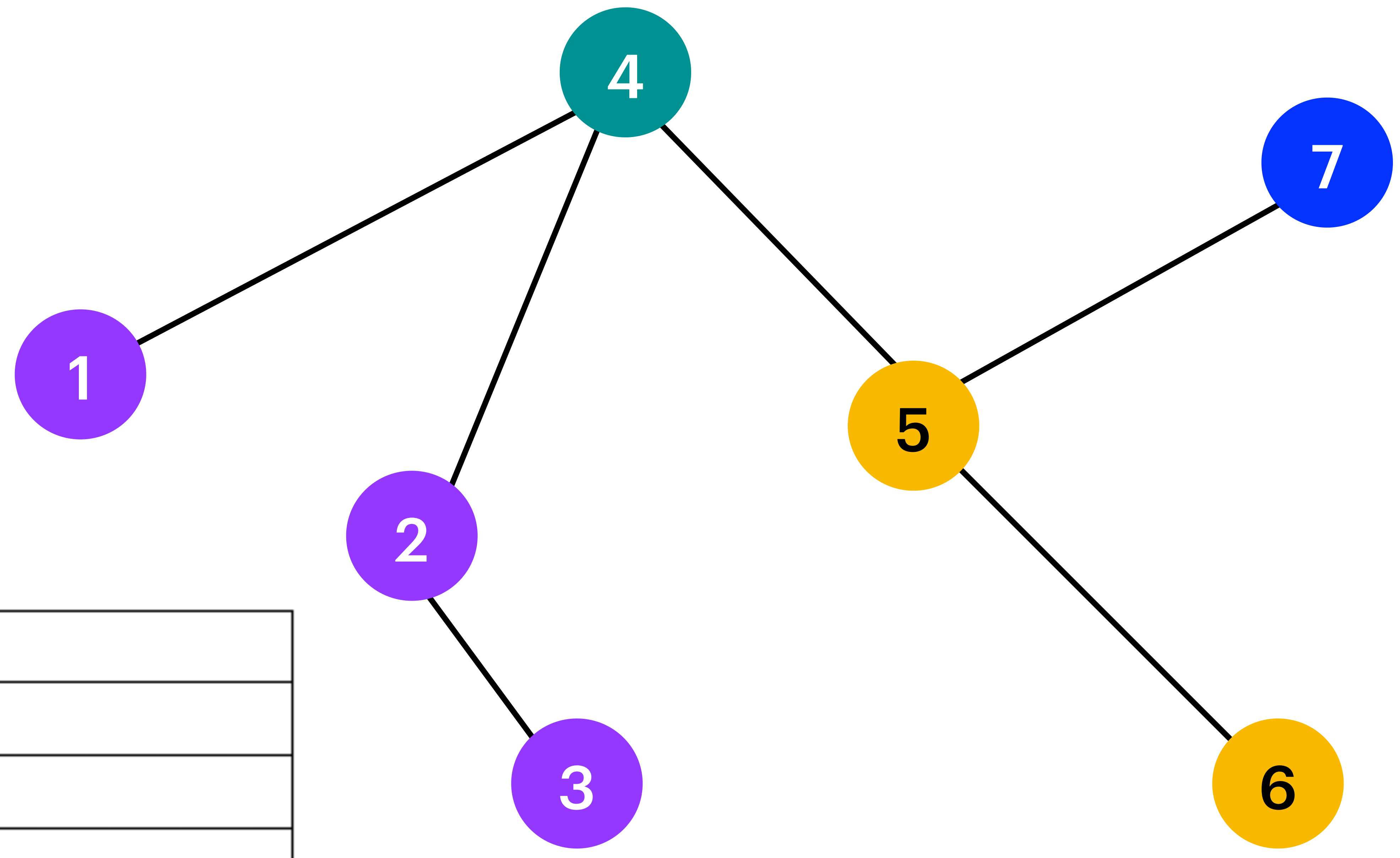

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$P_1(1)$	{{1 , 2}}
$P_1(2)$	
$P_1(3)$	
$P_1(4)$	
$P_1(5)$	
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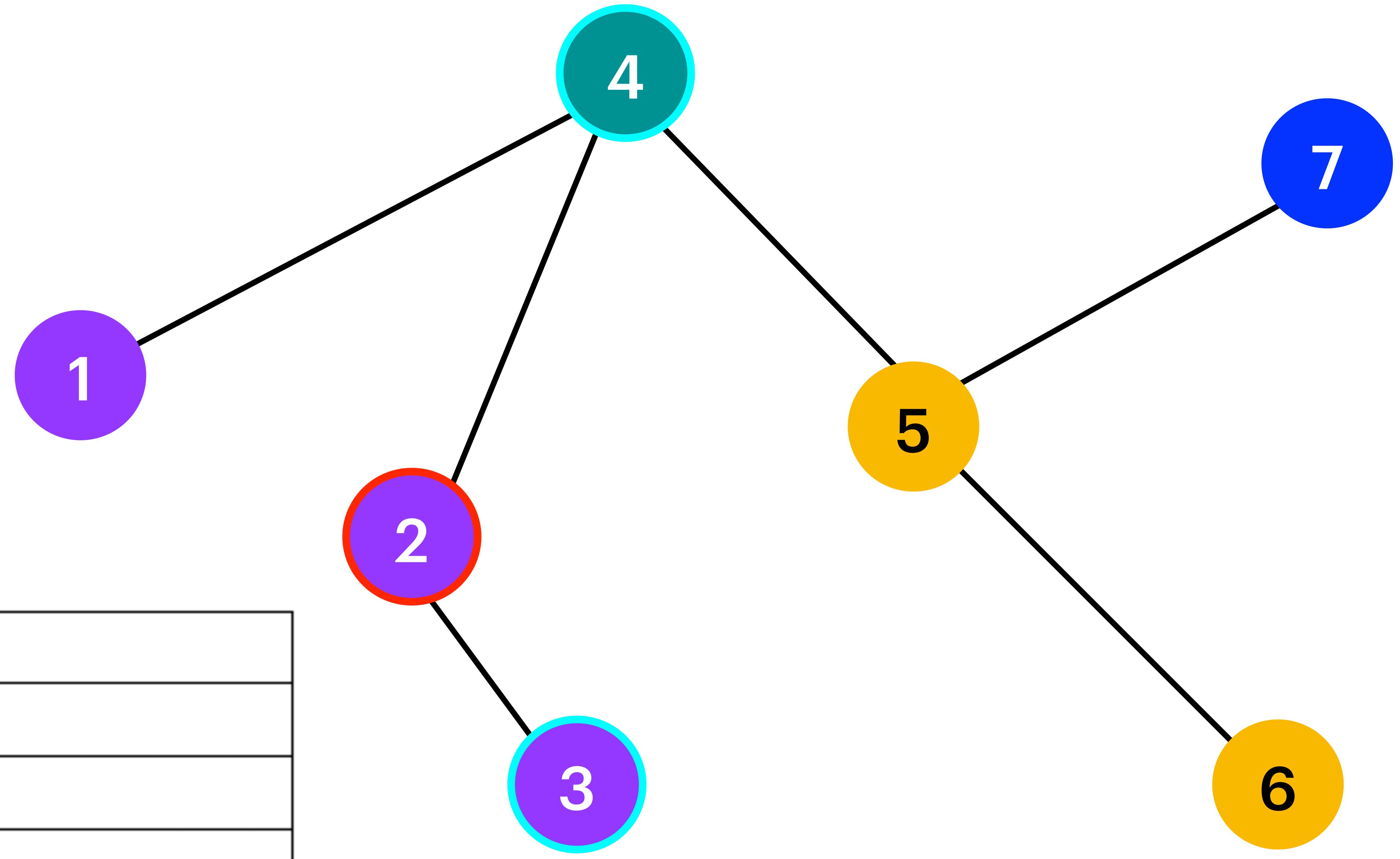
$P_1(7)$	

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}



γ 1 , 2 , 3 , 4

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P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	
$P_1(4)$	
$P_1(5)$	
$P_1(6)$	
$P_1(7)$	

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
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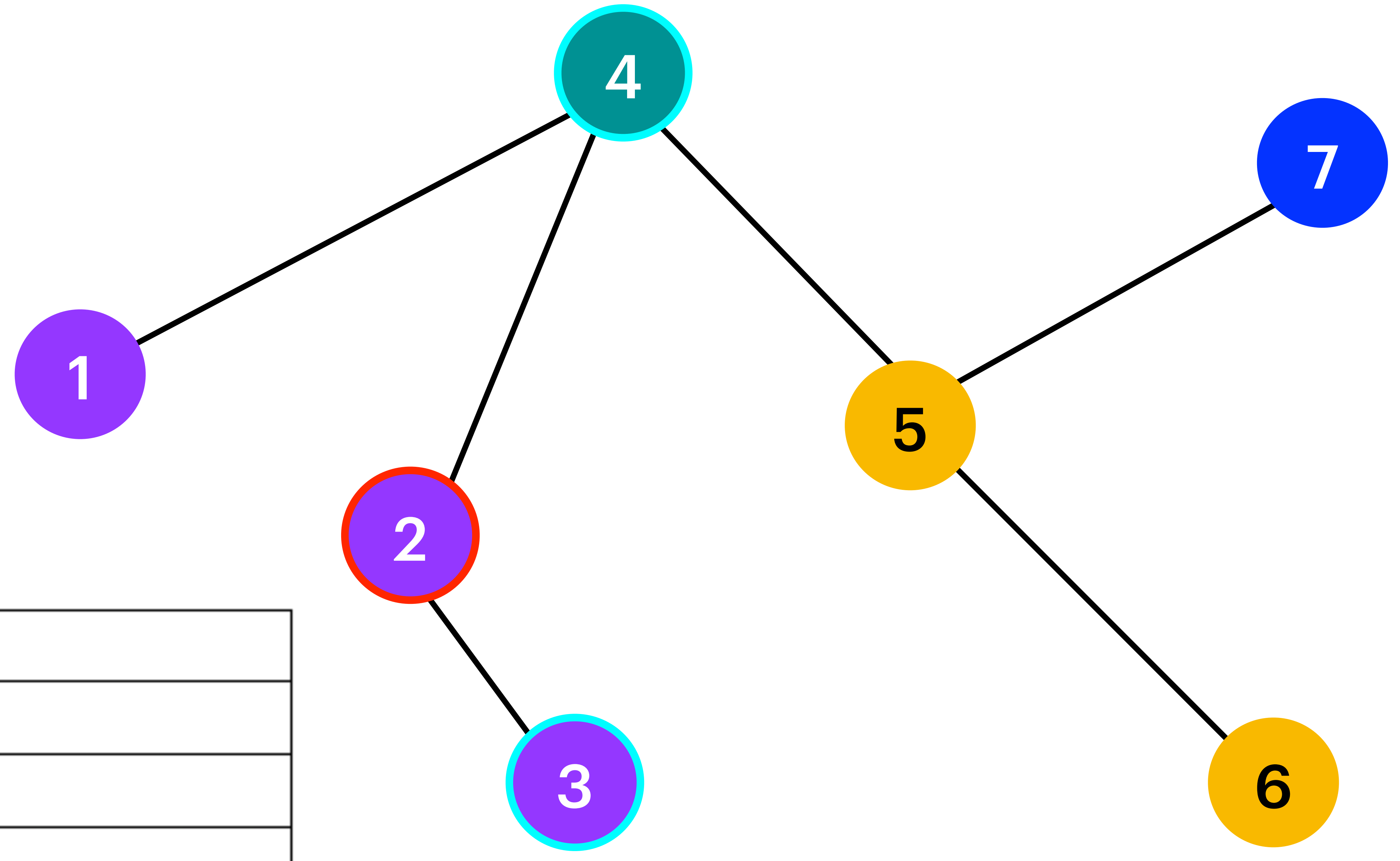

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P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	
$P_1(5)$	
$P_1(6)$	
$P_1(7)$	

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
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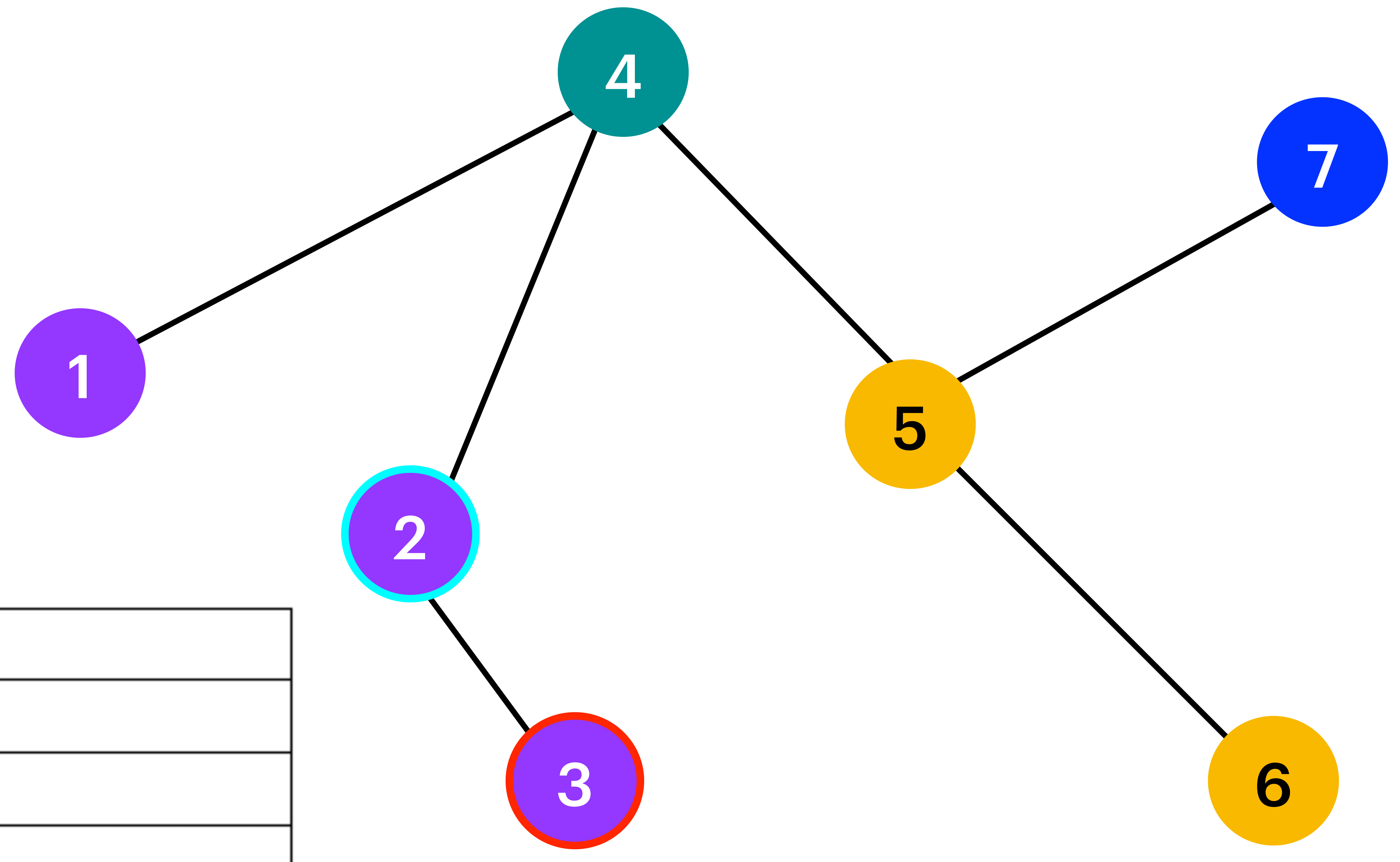

γ 1 , 2 , 3 , 4

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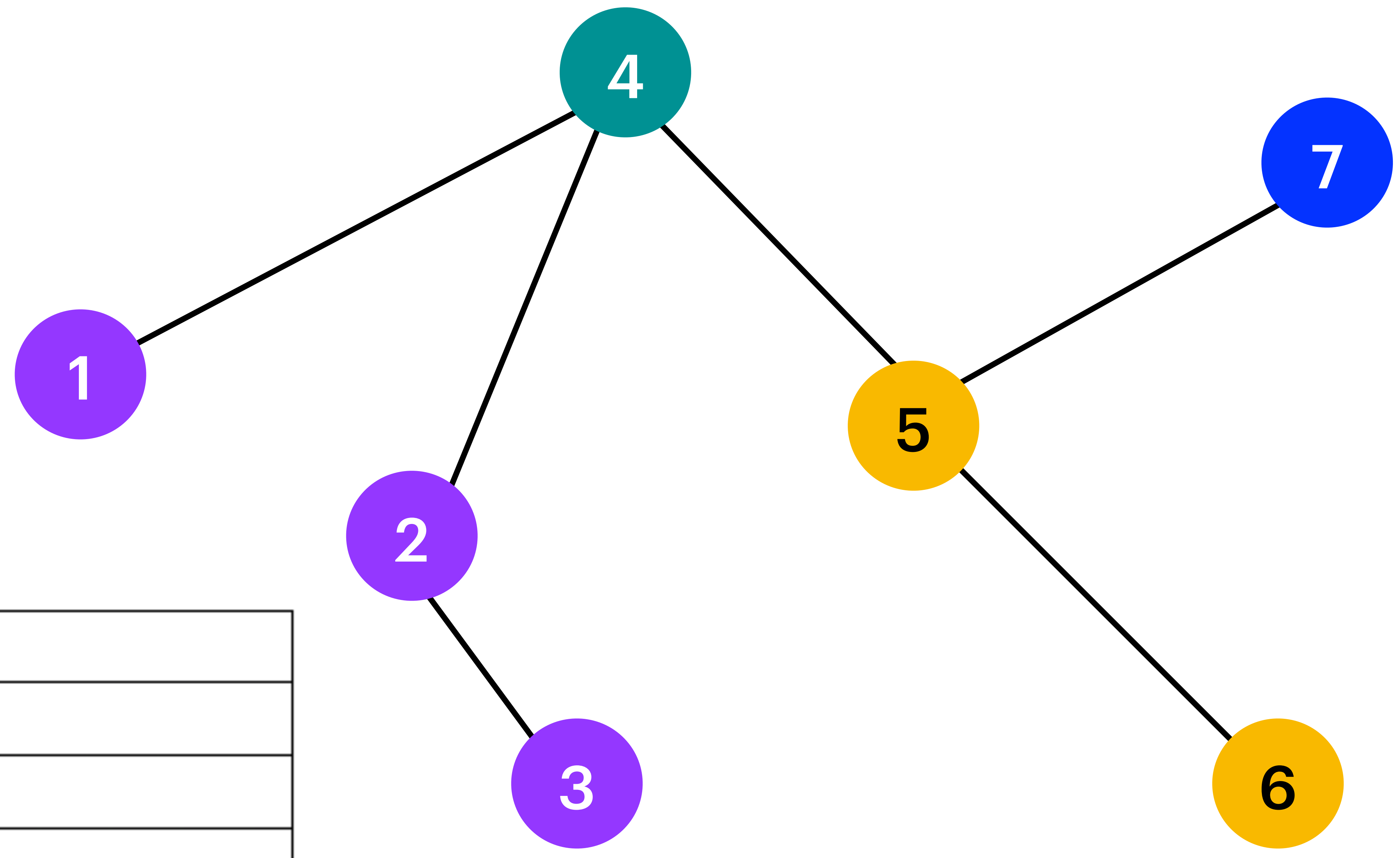
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	
$P_1(6)$	
$P_1(7)$	

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}



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$P_1(7)$	

P_0	
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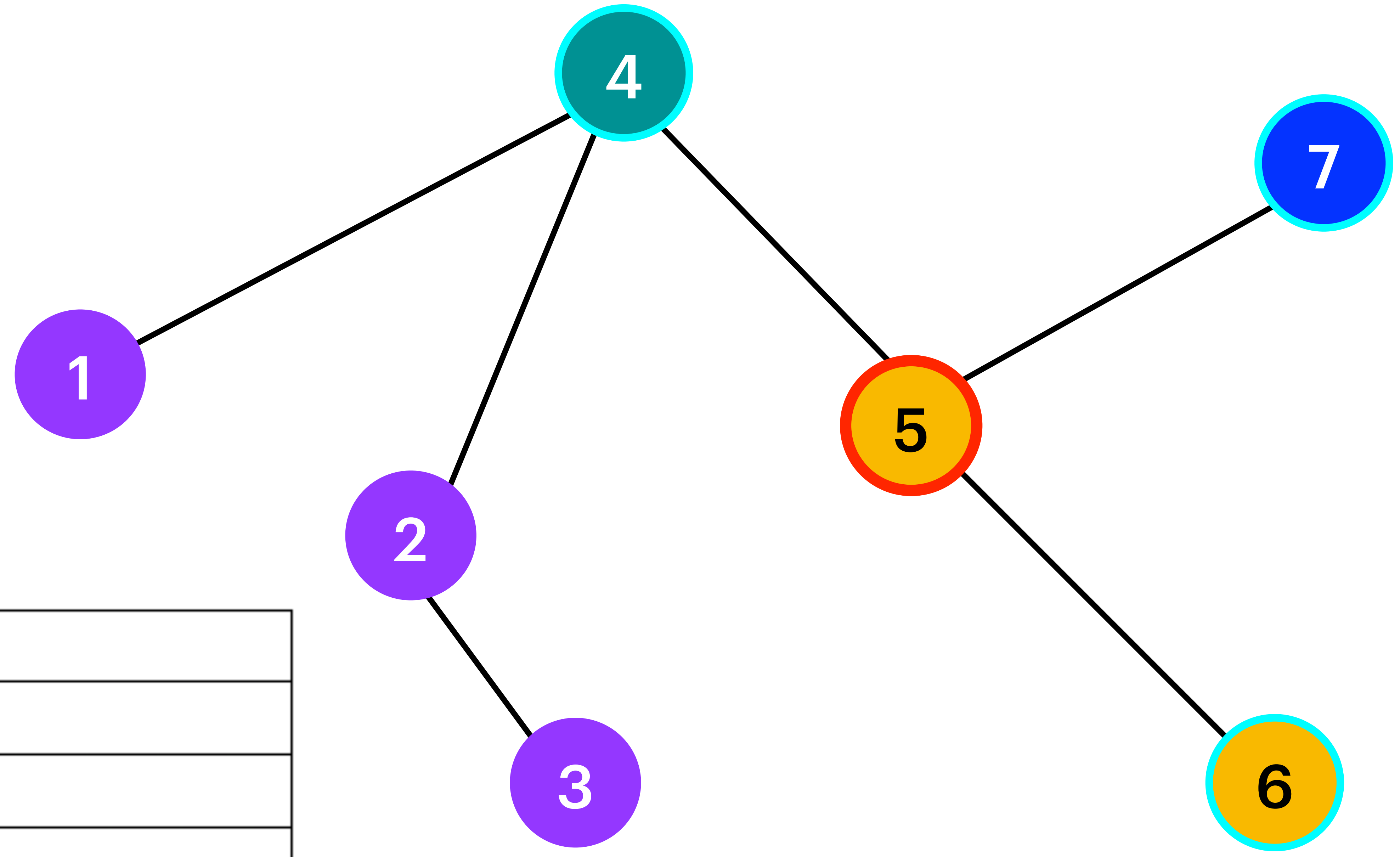
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}



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$P_1(7)$	

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P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
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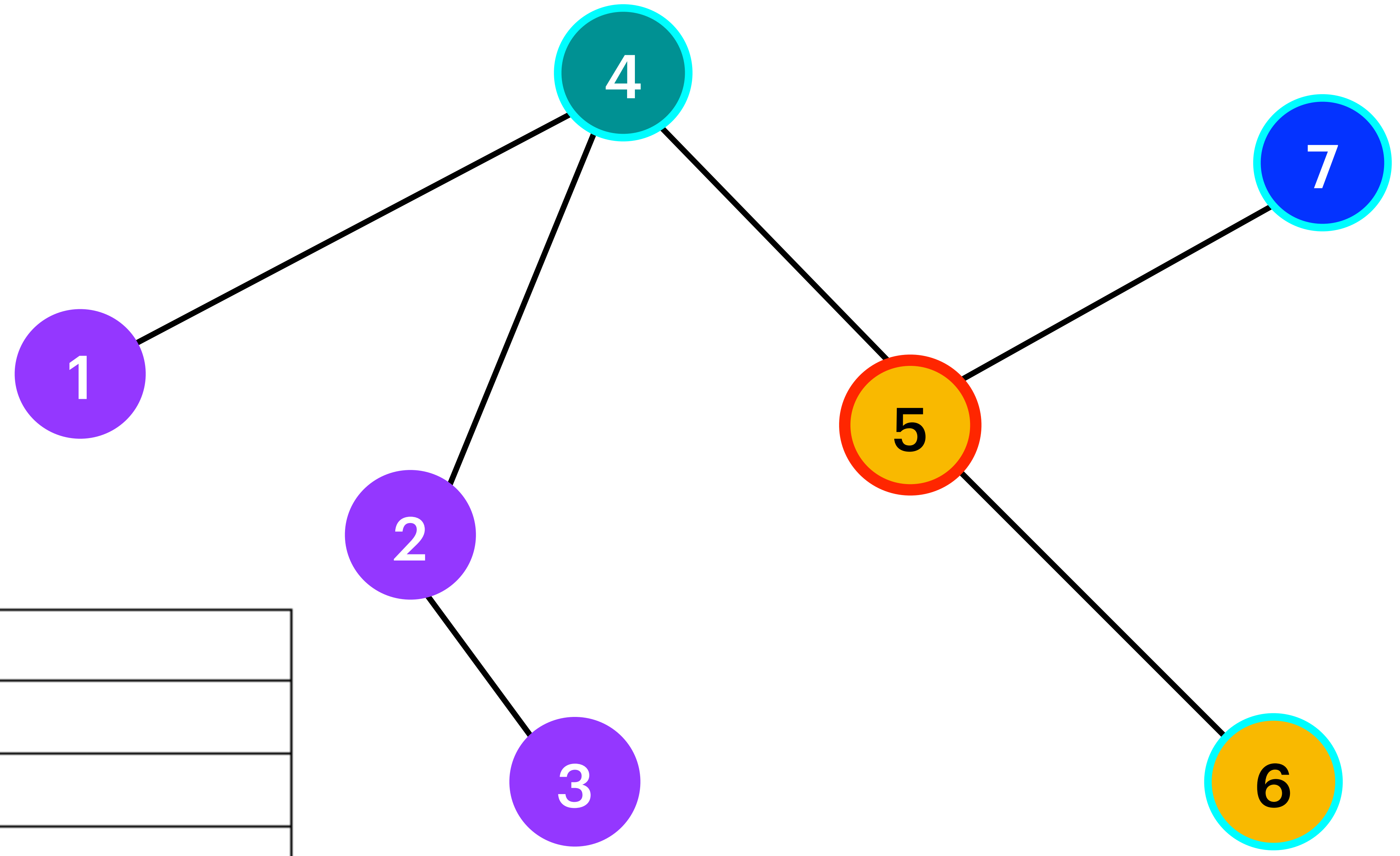

γ 1 , 2 , 3 , 4

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  - $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
- **for** $i = 1$ **to** $k - 1$ **do**
  - COLORFUL(G, i);
- **return** $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

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      - $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

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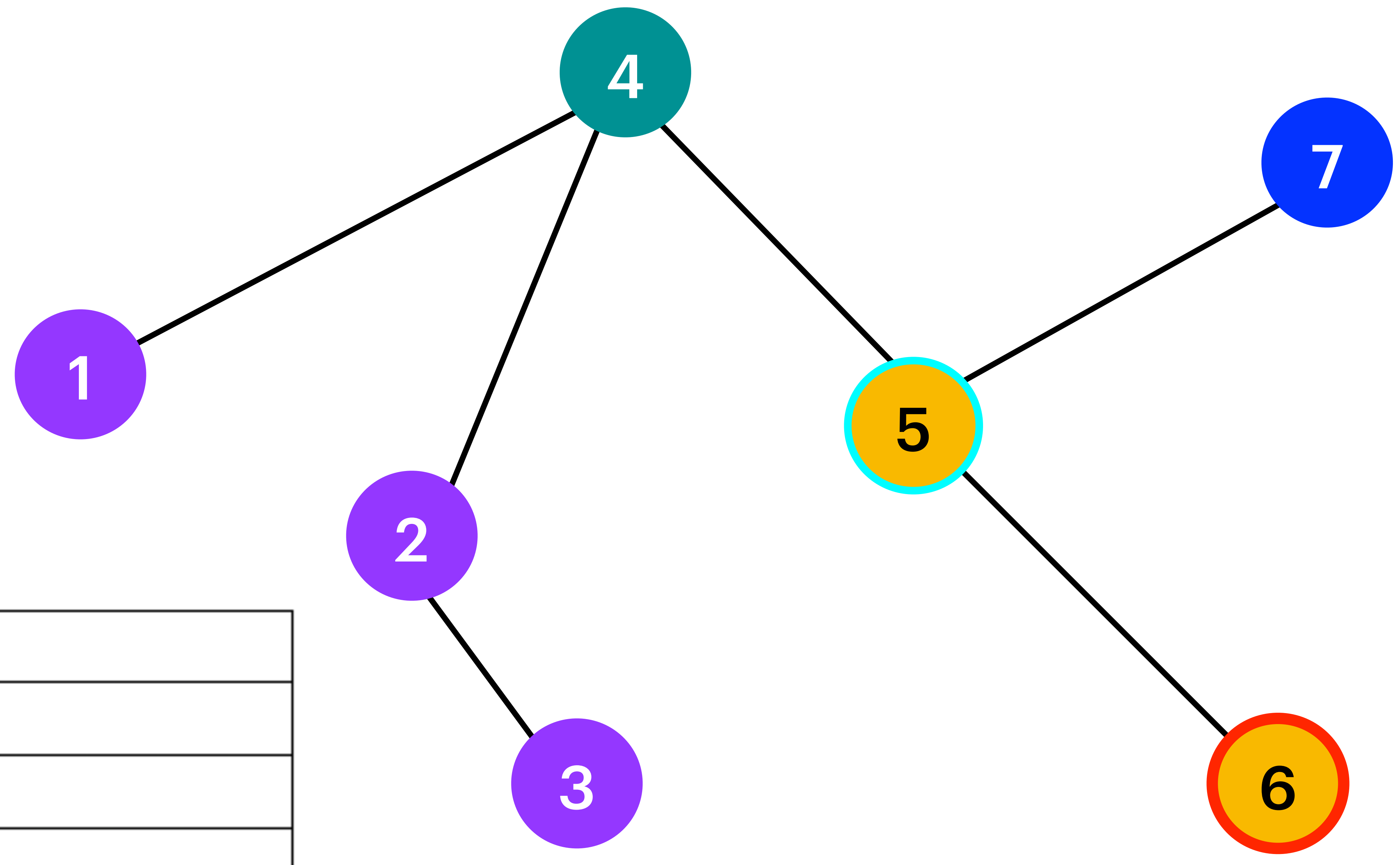
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	
$P_1(7)$	

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S



γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

Algorithm 2: RAINBOW(G, γ)

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  COLORFUL(G, i);
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

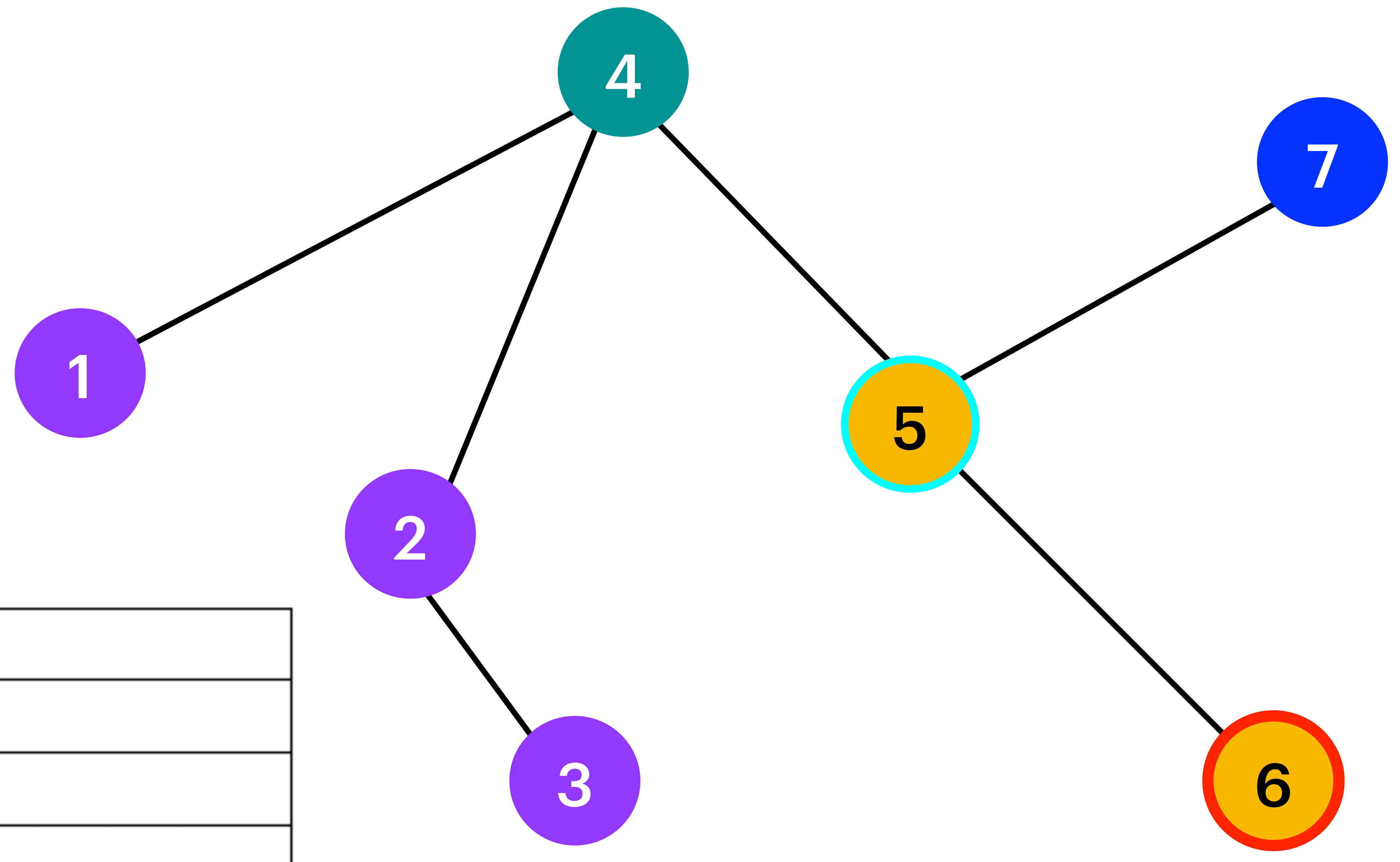
Algorithm 1: COLORFUL(G, i)

forall $v \in V$ **do**
  $P_i(v) \leftarrow \emptyset$;
  forall $x \in N(v)$ **do**
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}



γ 1 , 2 , 3 , 4

Colorful Paths

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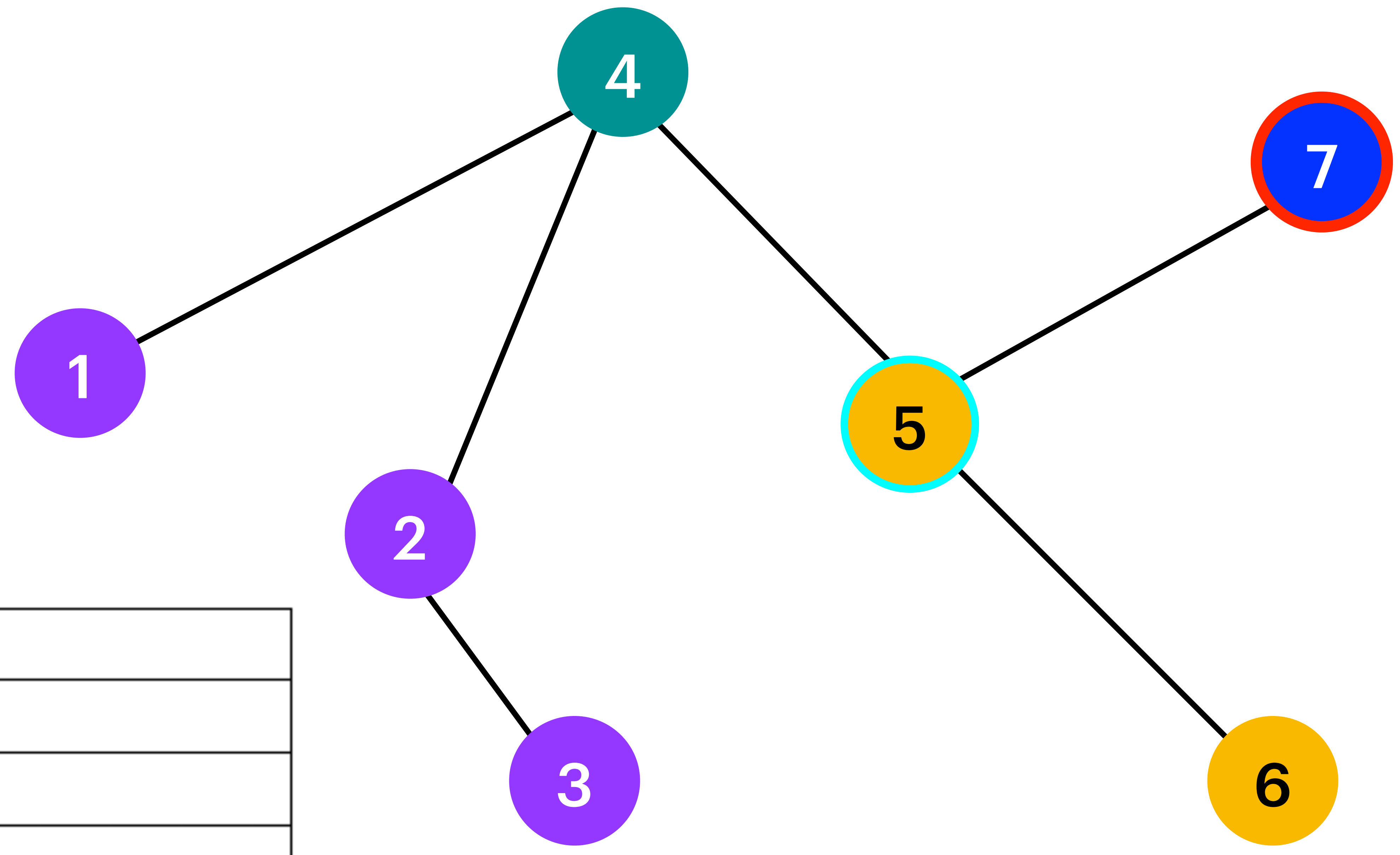


P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

P_0	
$P_0(1)$	{{1}}
$P_0(2)$	{{1}}
$P_0(3)$	{{1}}
$P_0(4)$	{{2}}
$P_0(5)$	{{3}
$P_0(6)$	{{3}}
$P_0(7)$	{{4}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

Algorithm 2: RAINBOW(G, γ)

```
forall v ∈ V do
    P_0(v) ← {{γ(v)}};
for i = 1 to k − 1 do
    COLORFUL(G, i);
return ⋃_{v∈V} P_{k−1}(v) ≠ ∅;
```

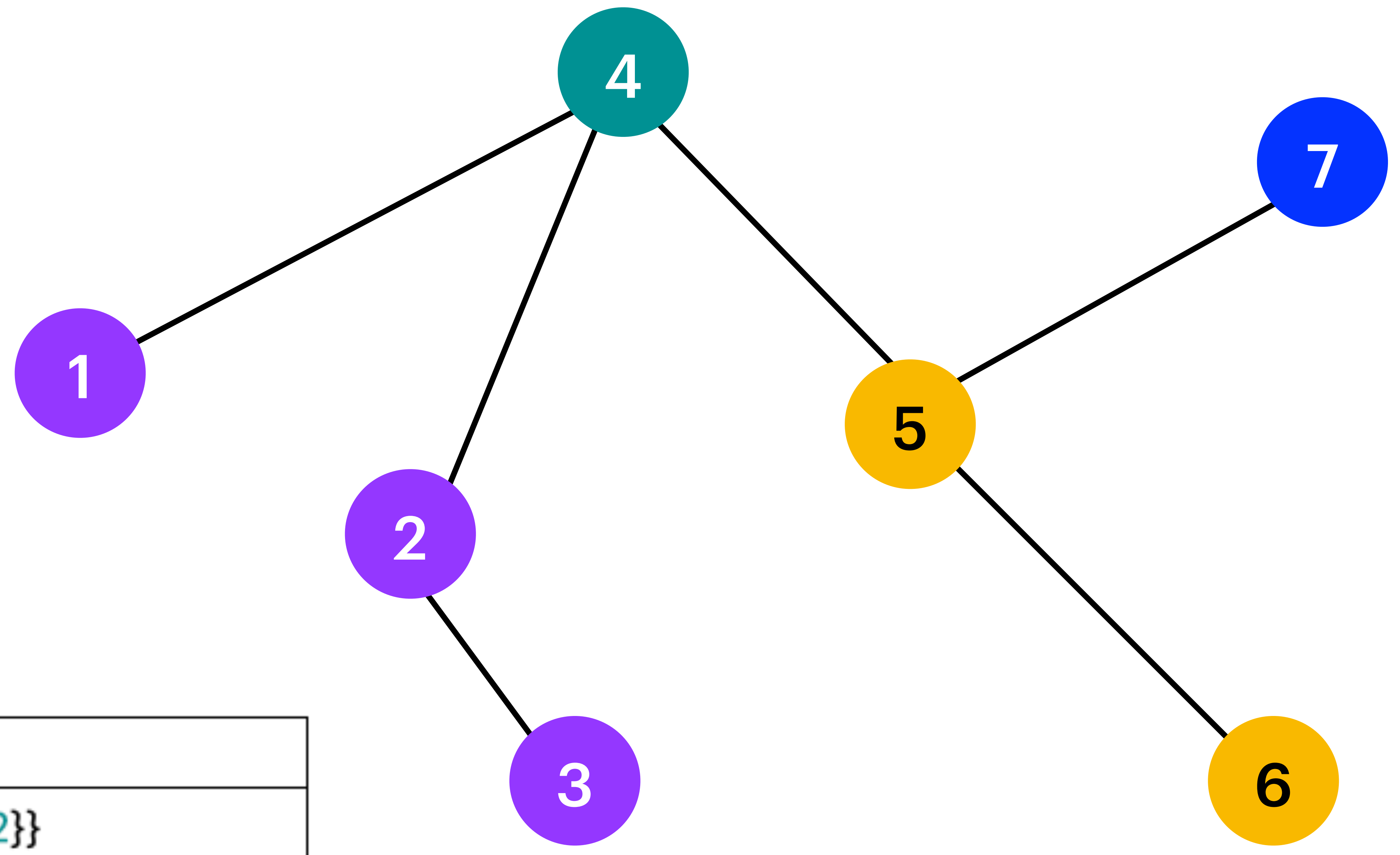
Algorithm 1: COLORFUL(G, i)

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i−1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

P_2	
$P_2(1)$	
$P_2(2)$	
$P_2(3)$	
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}



γ 1 , 2 , 3 , 4

Colorful Paths
Algorithm

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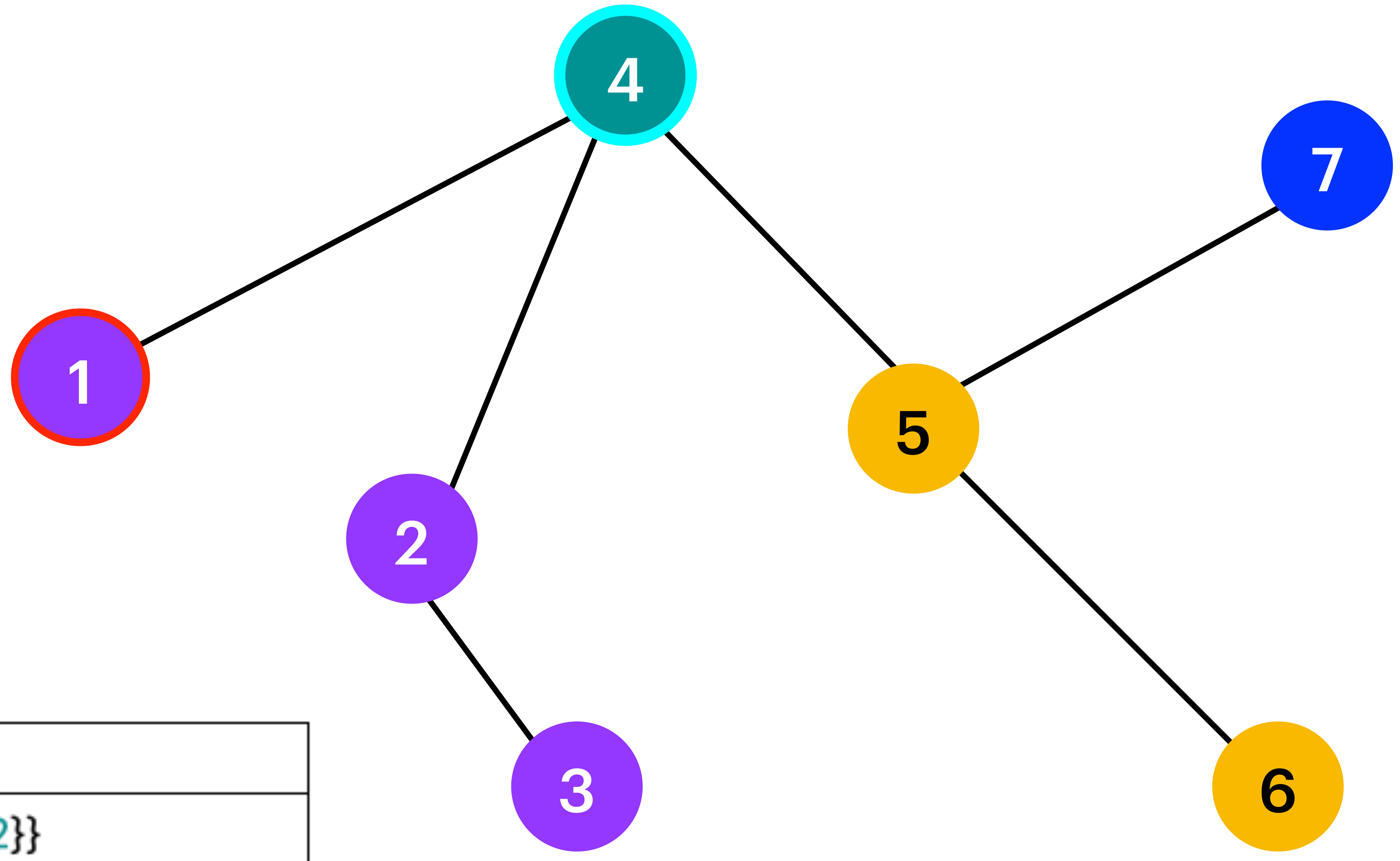
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_2	
$P_2(1)$	
$P_2(2)$	
$P_2(3)$	
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths
Algorithm

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```
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for i = 1 to k − 1 do
    COLORFUL(G, i);
return ⋃_{v∈V} P_{k−1}(v) ≠ ∅;
```

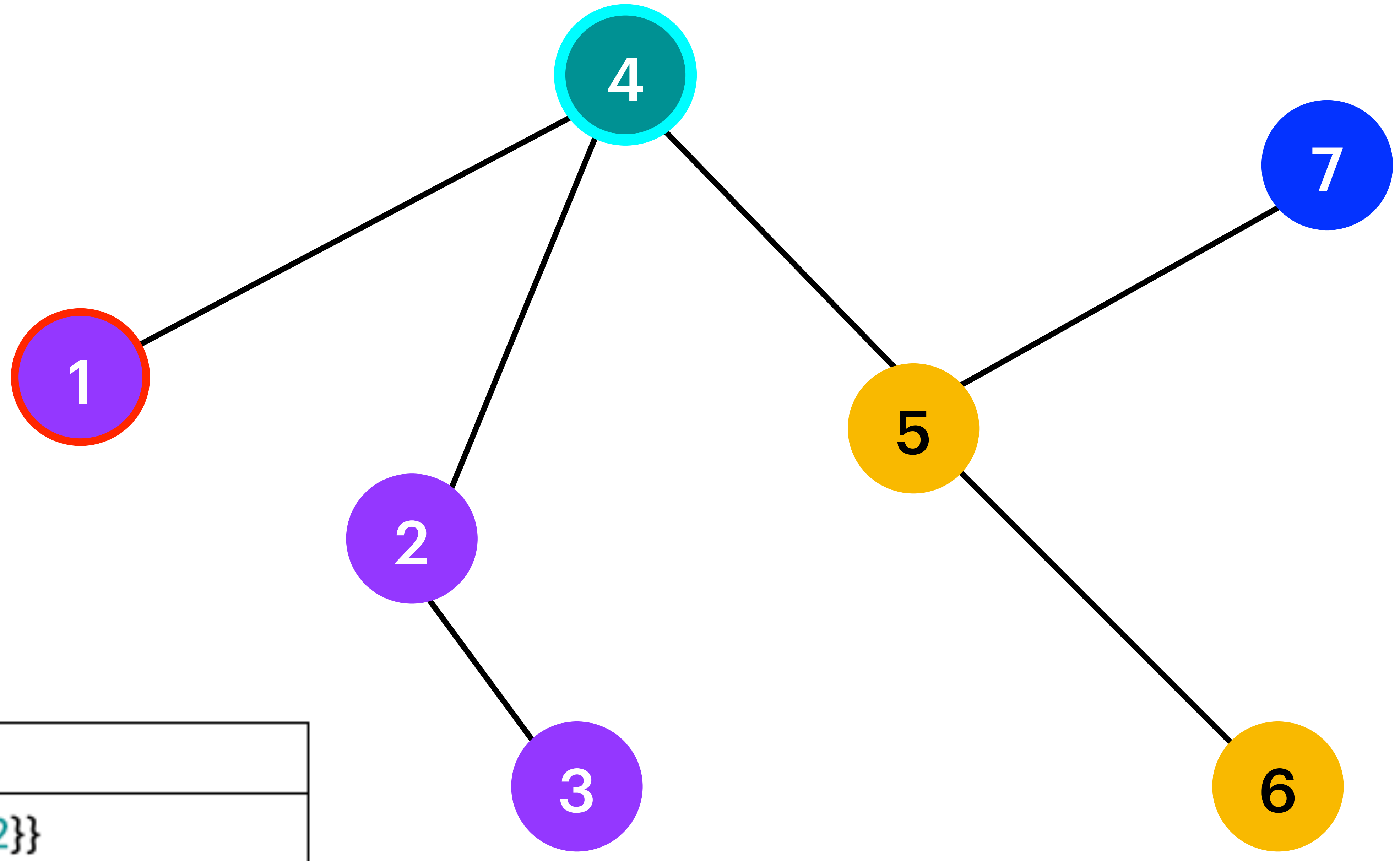
Algorithm 1: COLORFUL(G, i)

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i−1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	
$P_2(3)$	
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}



γ 1 , 2 , 3 , 4

Colorful Paths
Algorithm

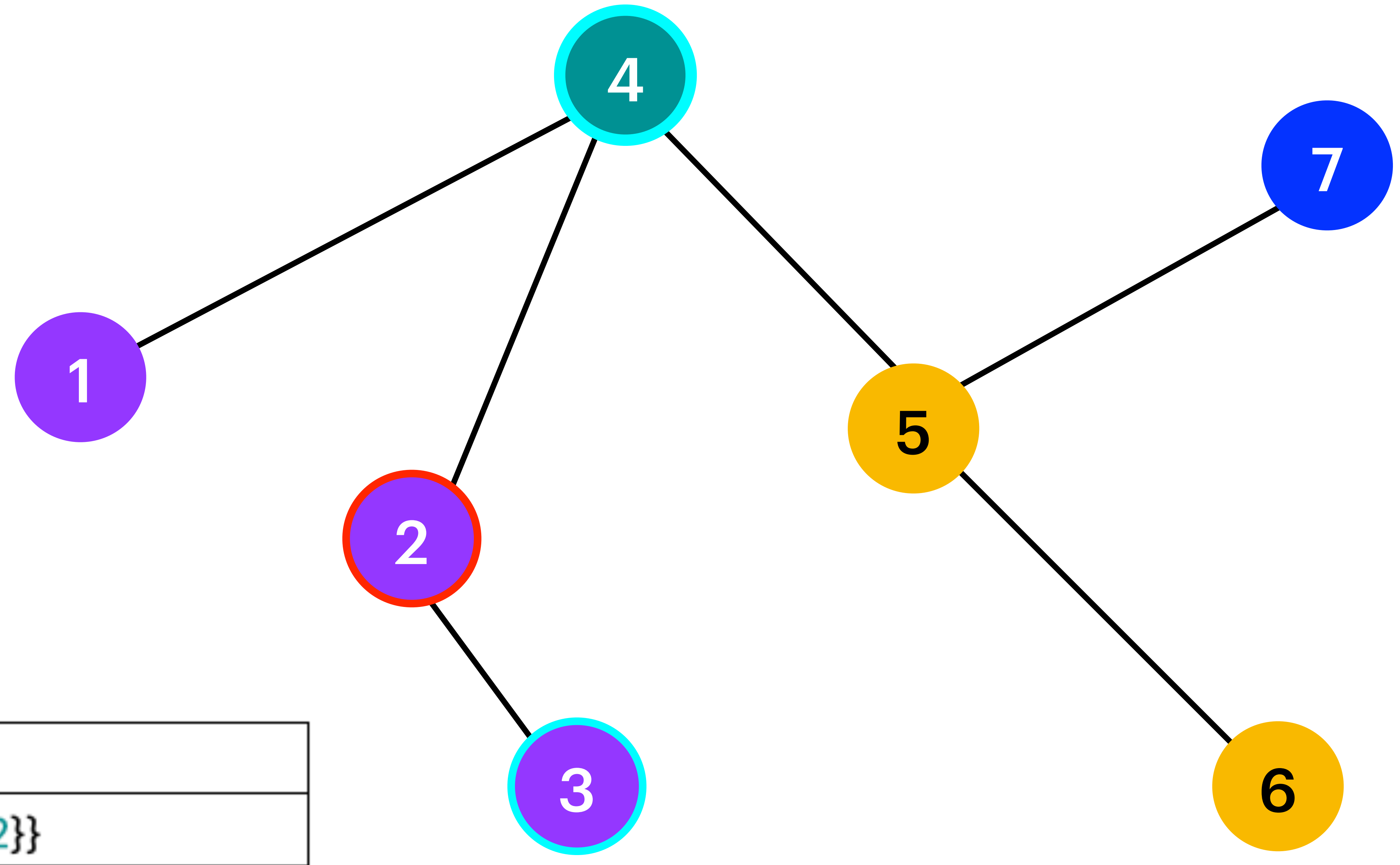
Algorithm 2: $\textsc{Rainbow}(G, \gamma)$

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  $\textsc{Colorful}(G, i)$;
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: $\textsc{Colorful}(G, i)$

forall $v \in V$ **do**
  $P_i(v) \leftarrow \emptyset$;
  forall $x \in N(v)$ **do**
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	
$P_2(3)$	
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

Algorithm 2: $\textsc{Rainbow}(G, \gamma)$

```
forall v ∈ V do
    P_0(v) ← {{γ(v)}};
for i = 1 to k − 1 do
    COLORFUL(G, i);
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```
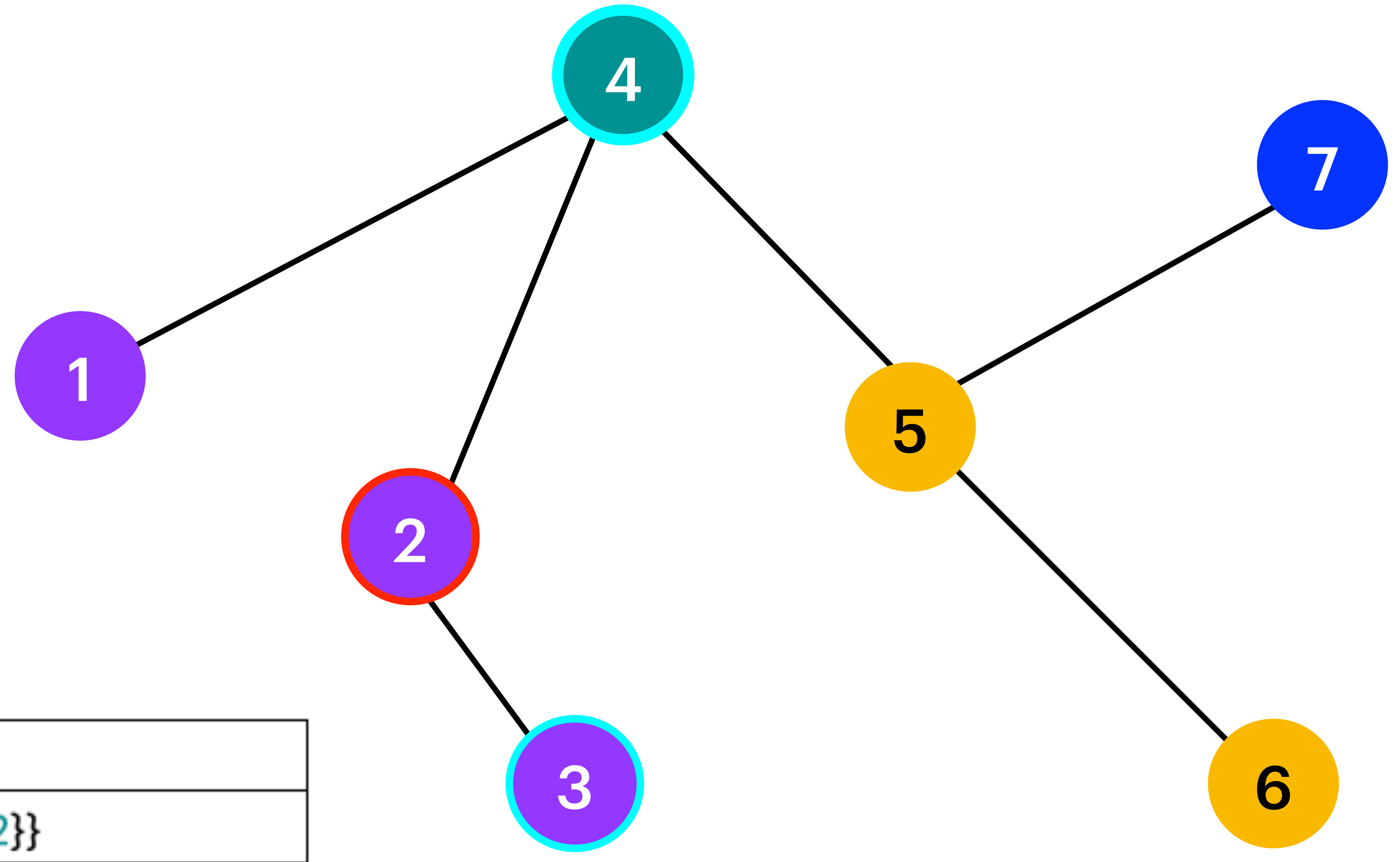

Algorithm 1: $\textsc{Colorful}(G, i)$

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i−1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
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$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
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$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

γ 1 , 2 , 3 , 4

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Colorful Paths

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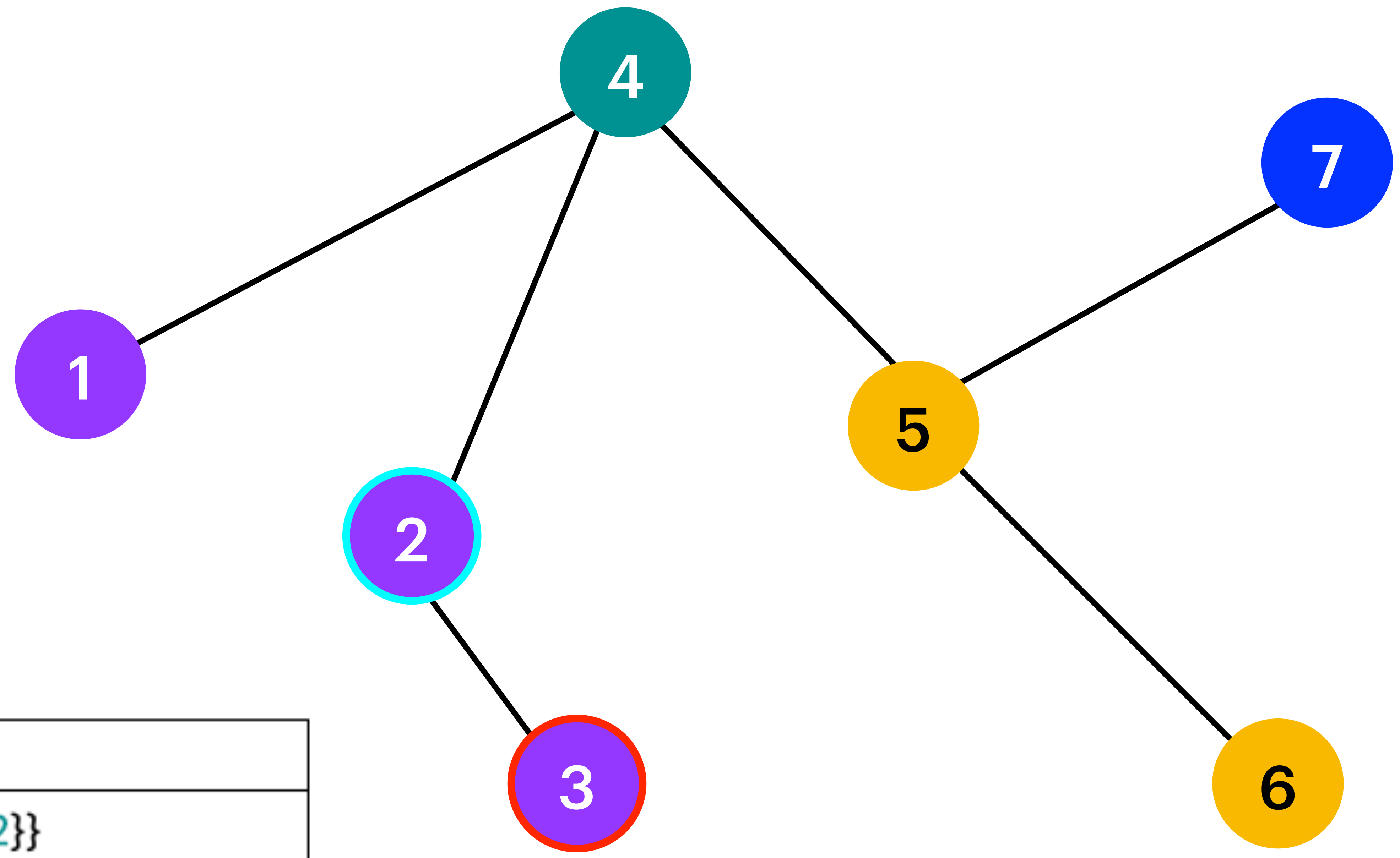
P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

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γ 1 , 2 , 3 , 4

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P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	

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$P_1(1)$	{{1 , 2}}
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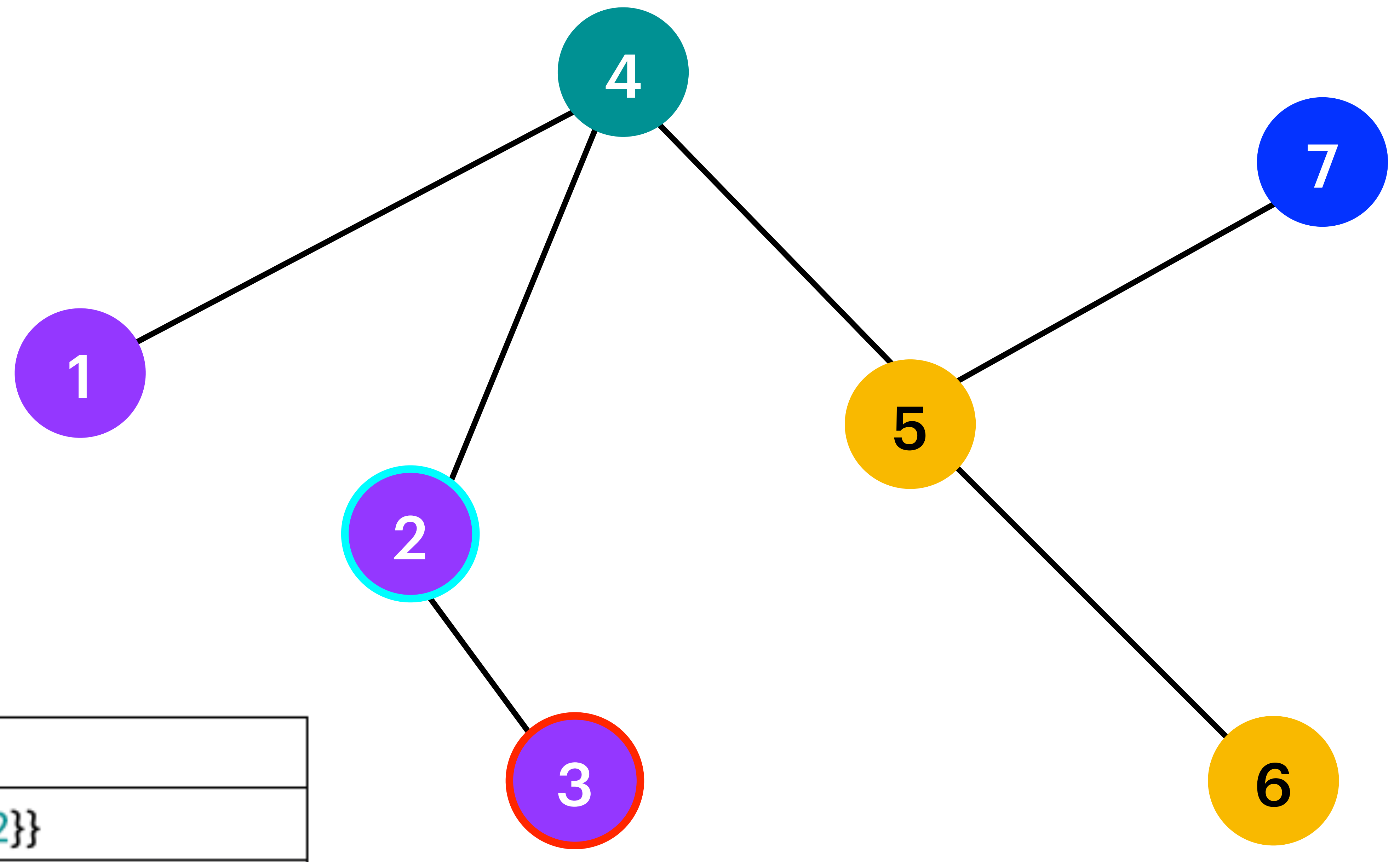

γ 1 , 2 , 3 , 4

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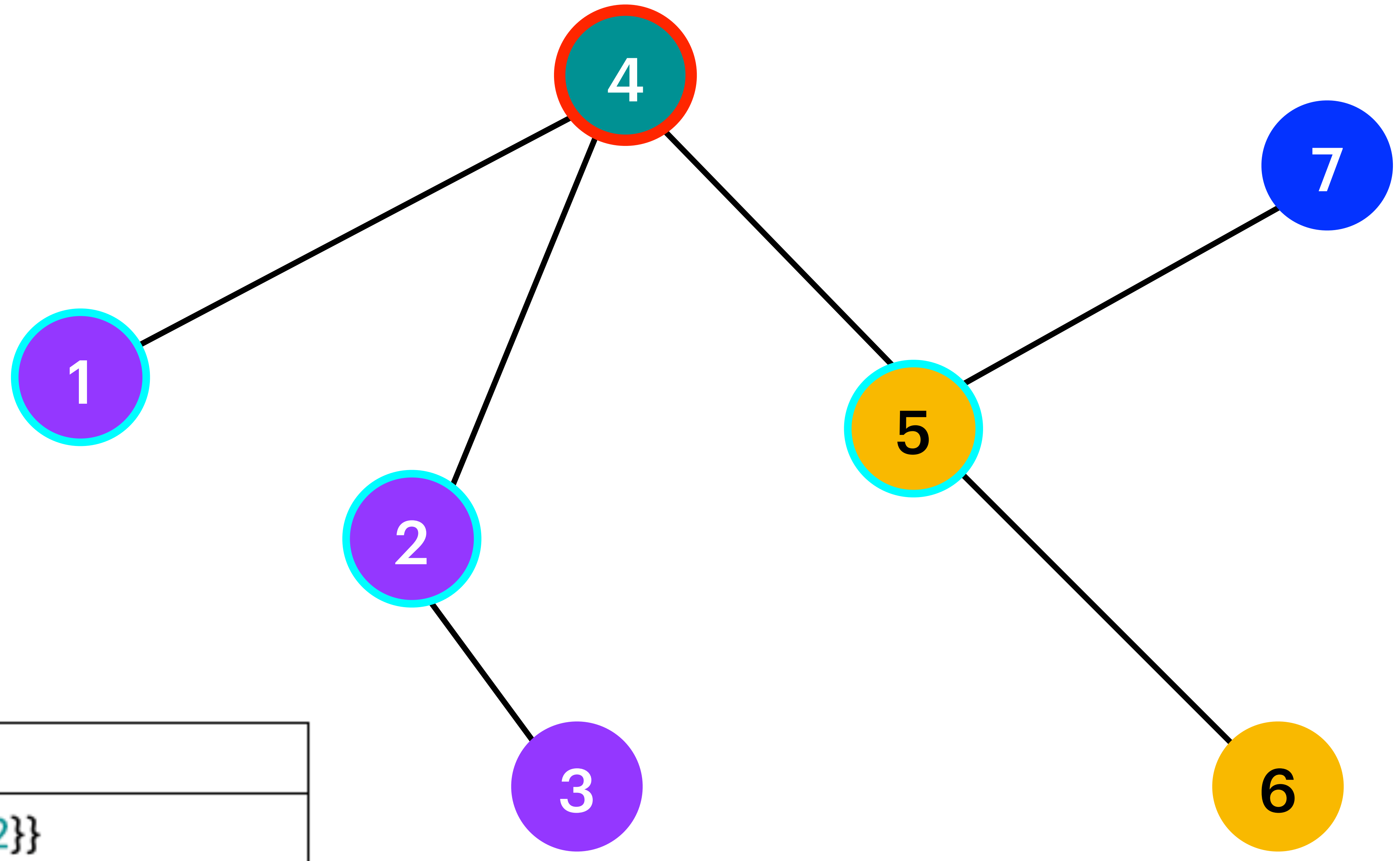
P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
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$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

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P_2	
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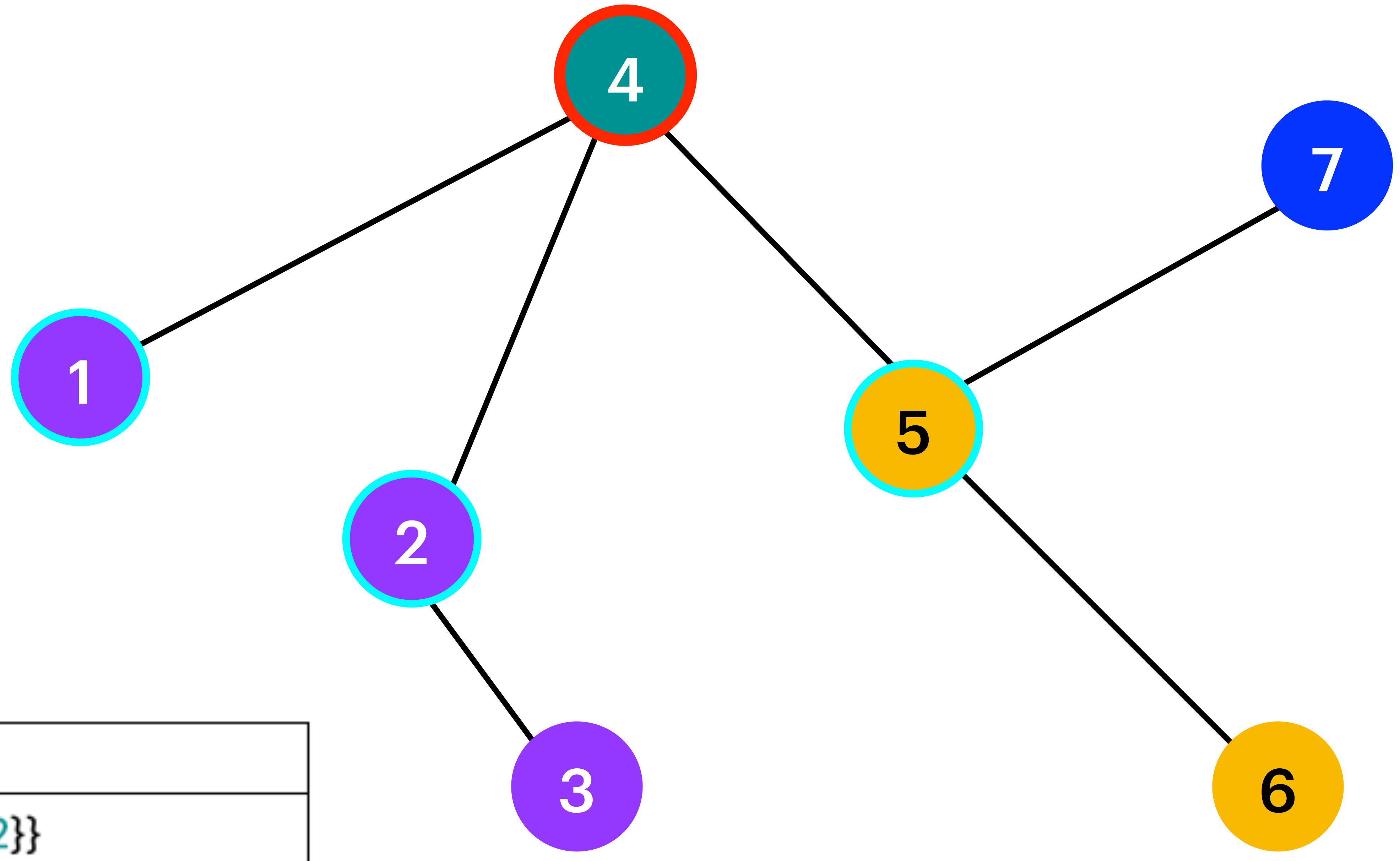
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	



P_1	
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P_2	
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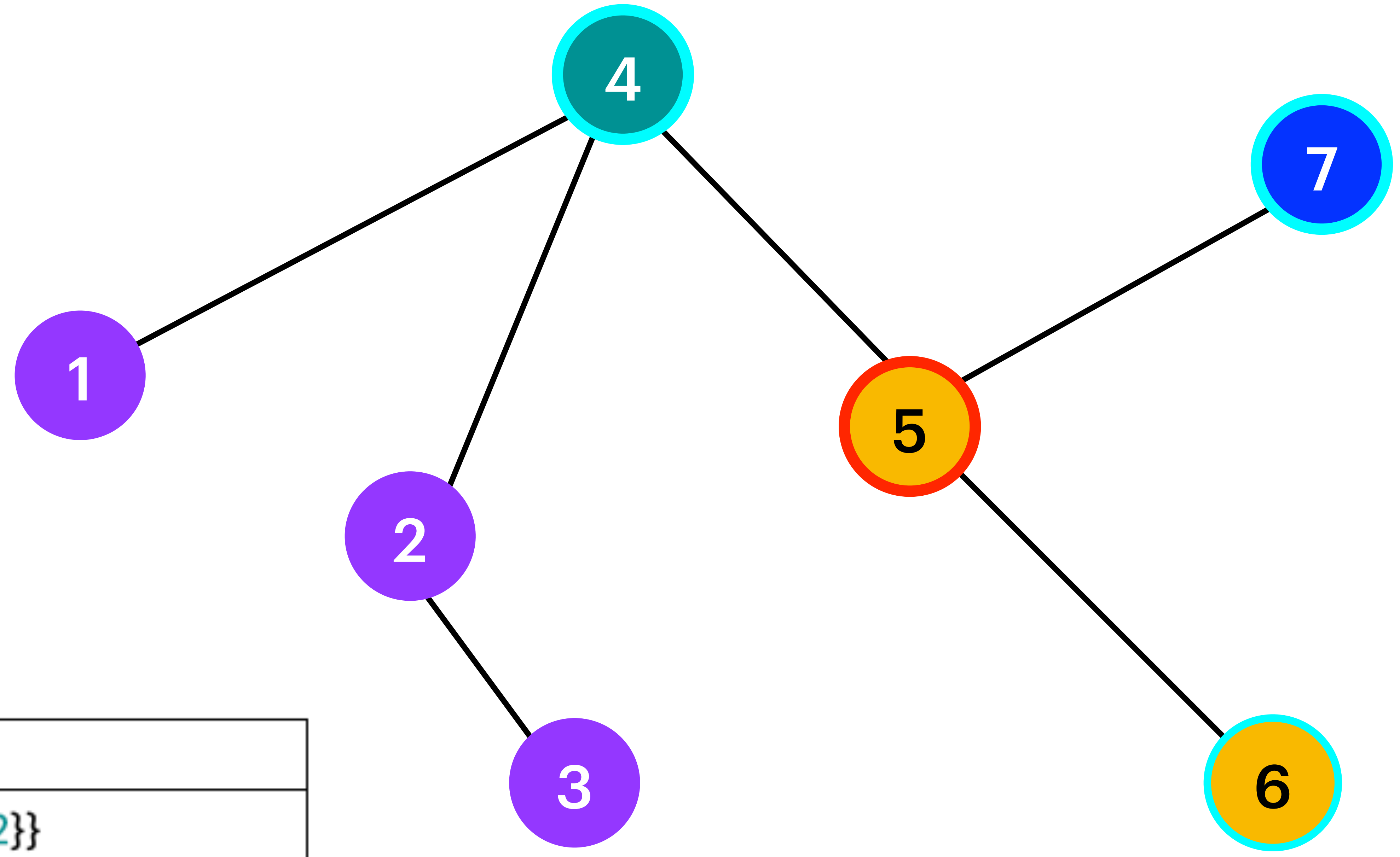
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	
$P_2(6)$	
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
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γ 1 , 2 , 3 , 4

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P_2	
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$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	
$P_2(7)$	

P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
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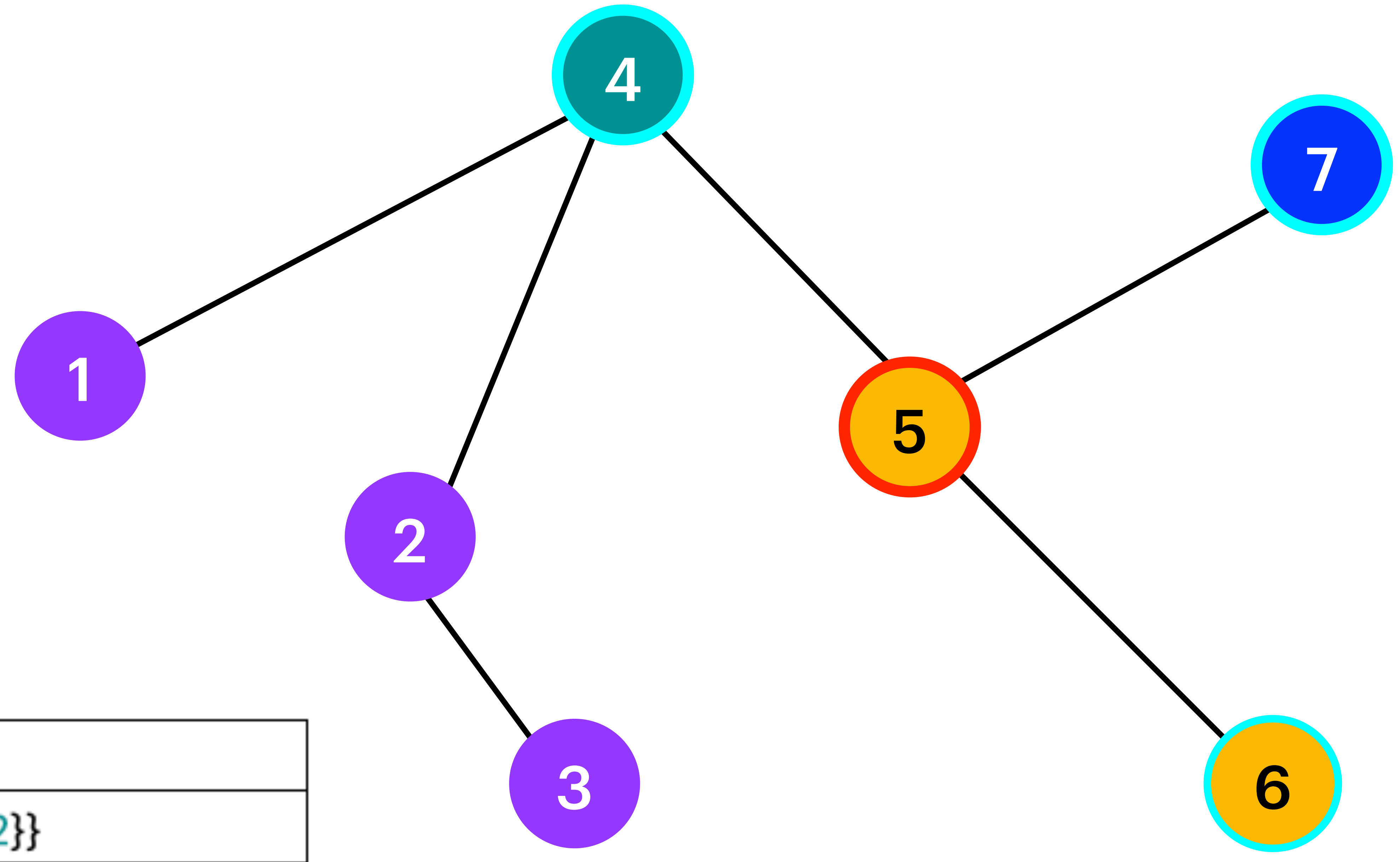
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}



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γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

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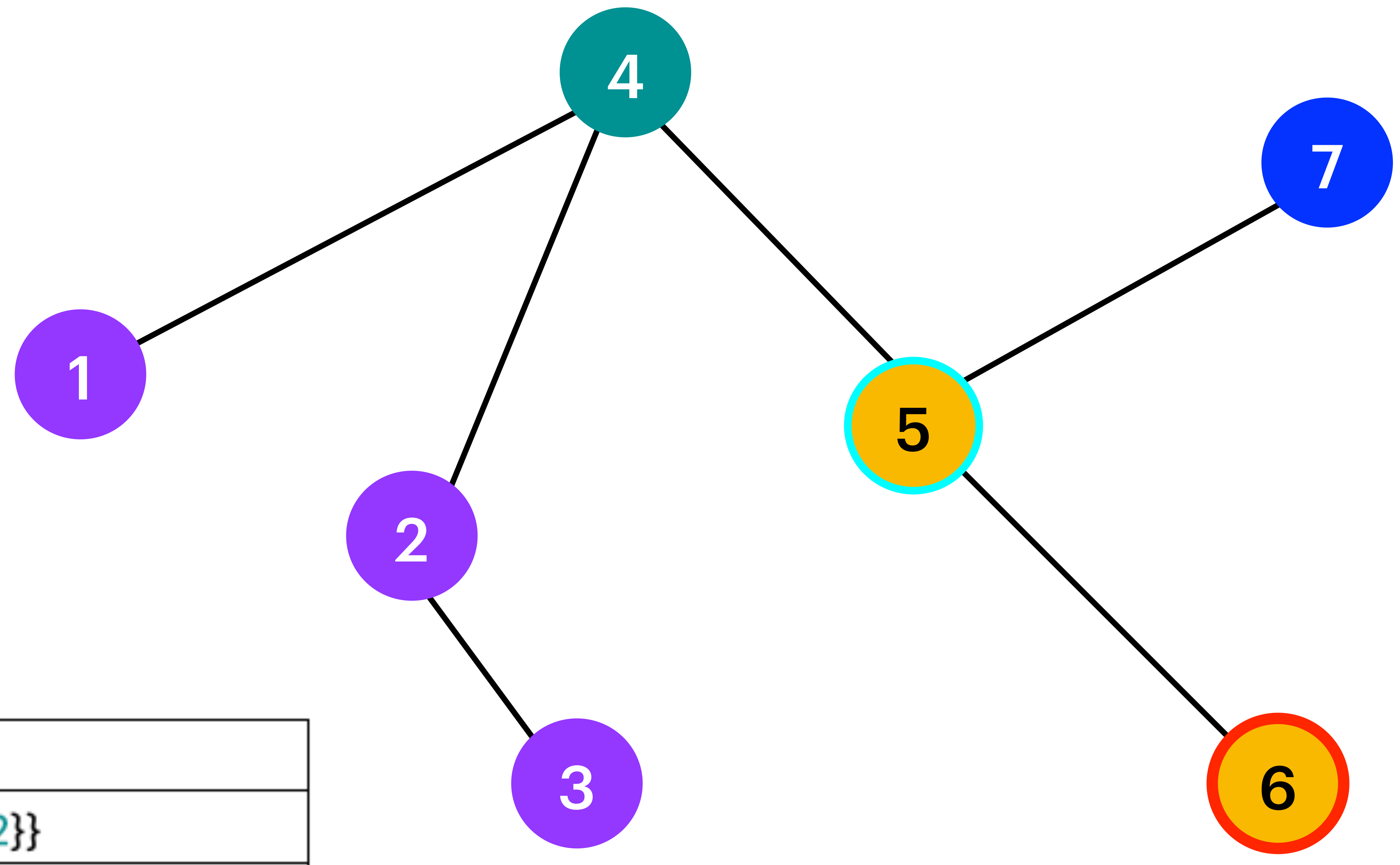
P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths
Algorithm

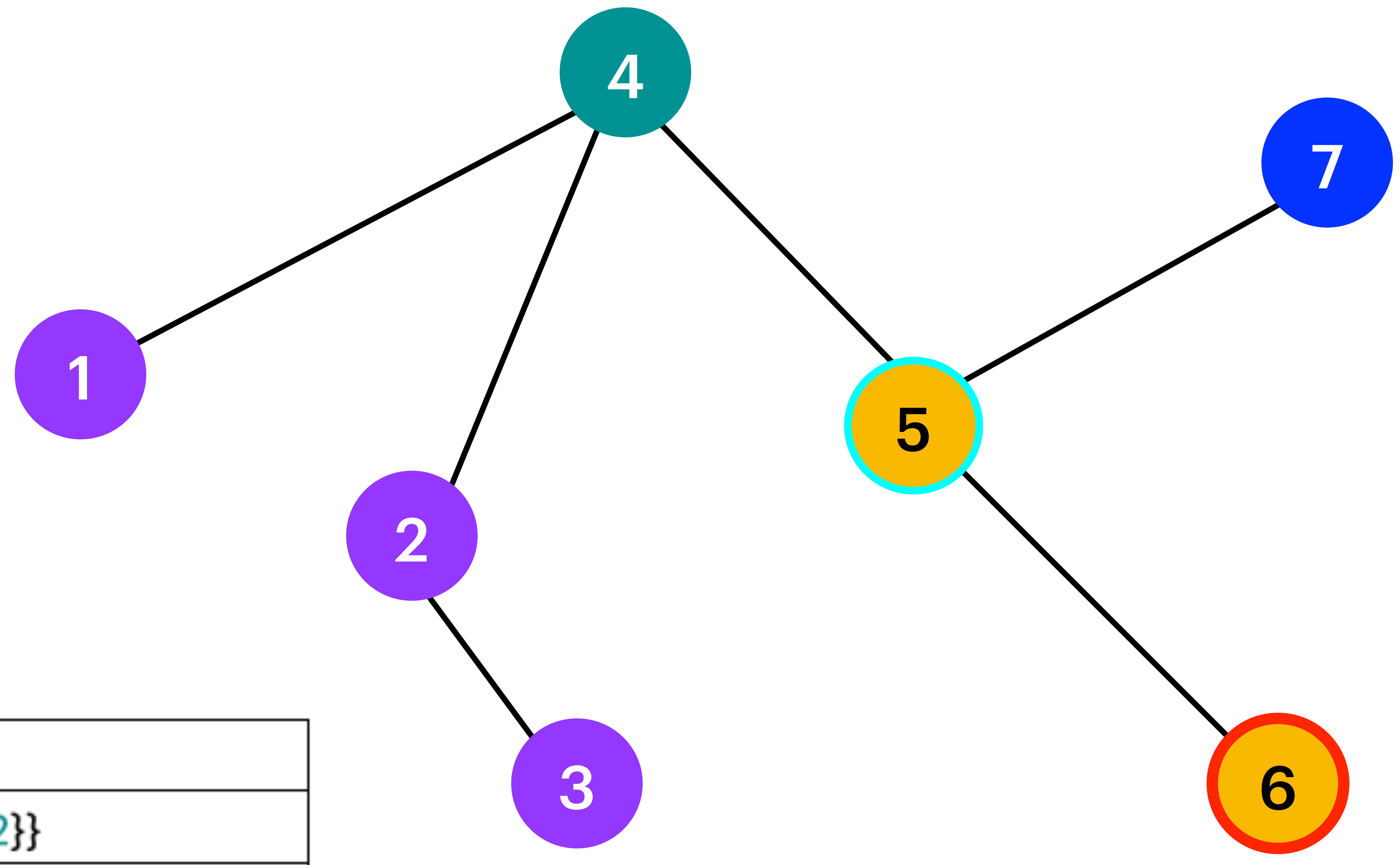
Algorithm 2: RAINBOW(G, γ)

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  COLORFUL(G, i);
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: COLORFUL(G, i)

forall $v \in V$ **do**
  $P_i(v) \leftarrow \emptyset$;
  forall $x \in N(v)$ **do**
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

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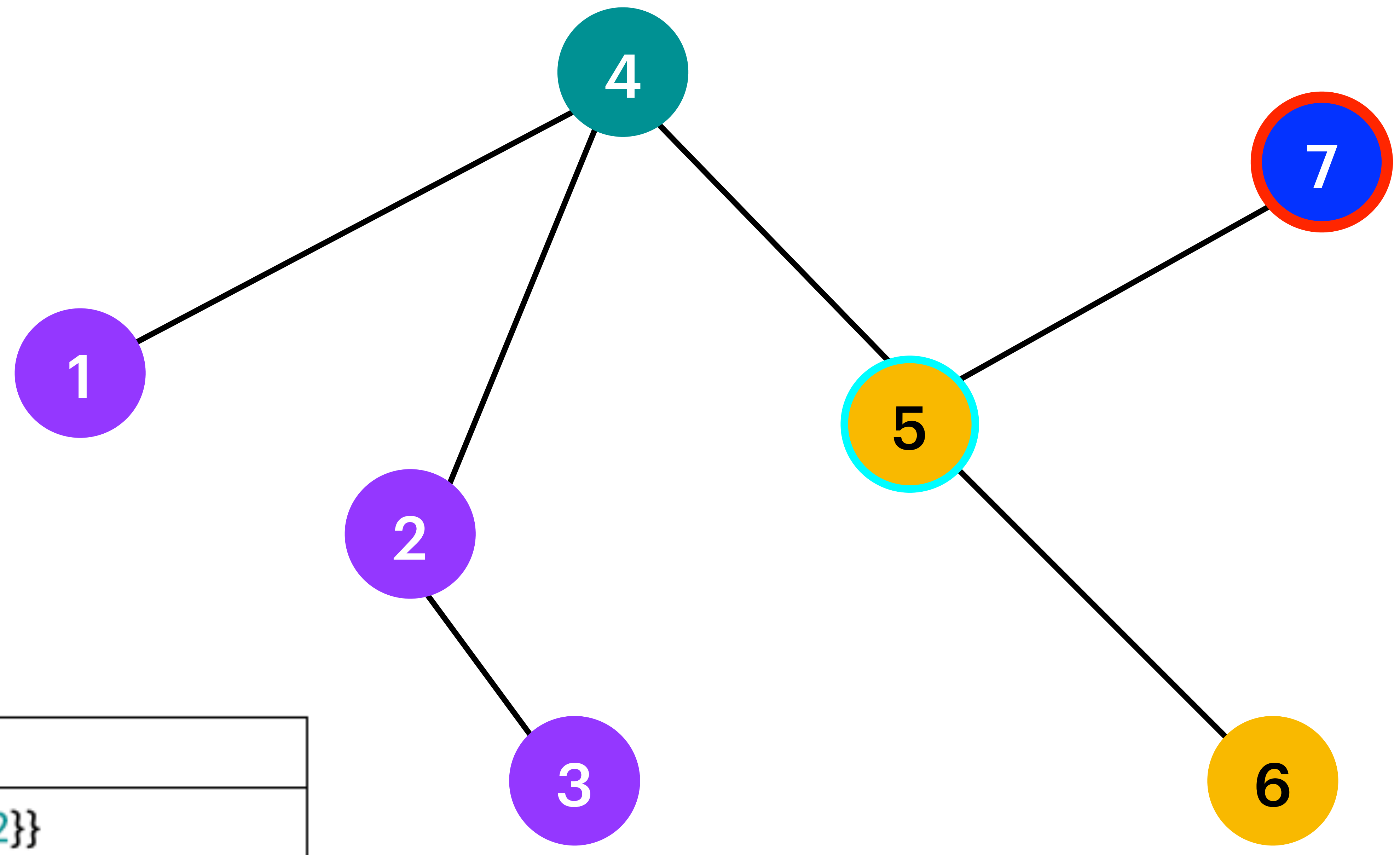
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}



P_1	
$P_1(1)$	{{1 , 2}}
$P_1(2)$	{{1 , 2}}
$P_1(3)$	∅
$P_1(4)$	{{1 , 2} , {2 , 3}}
$P_1(5)$	{{2 , 3} , {3 , 4}}
$P_1(6)$	∅
$P_1(7)$	{{3 , 4}}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

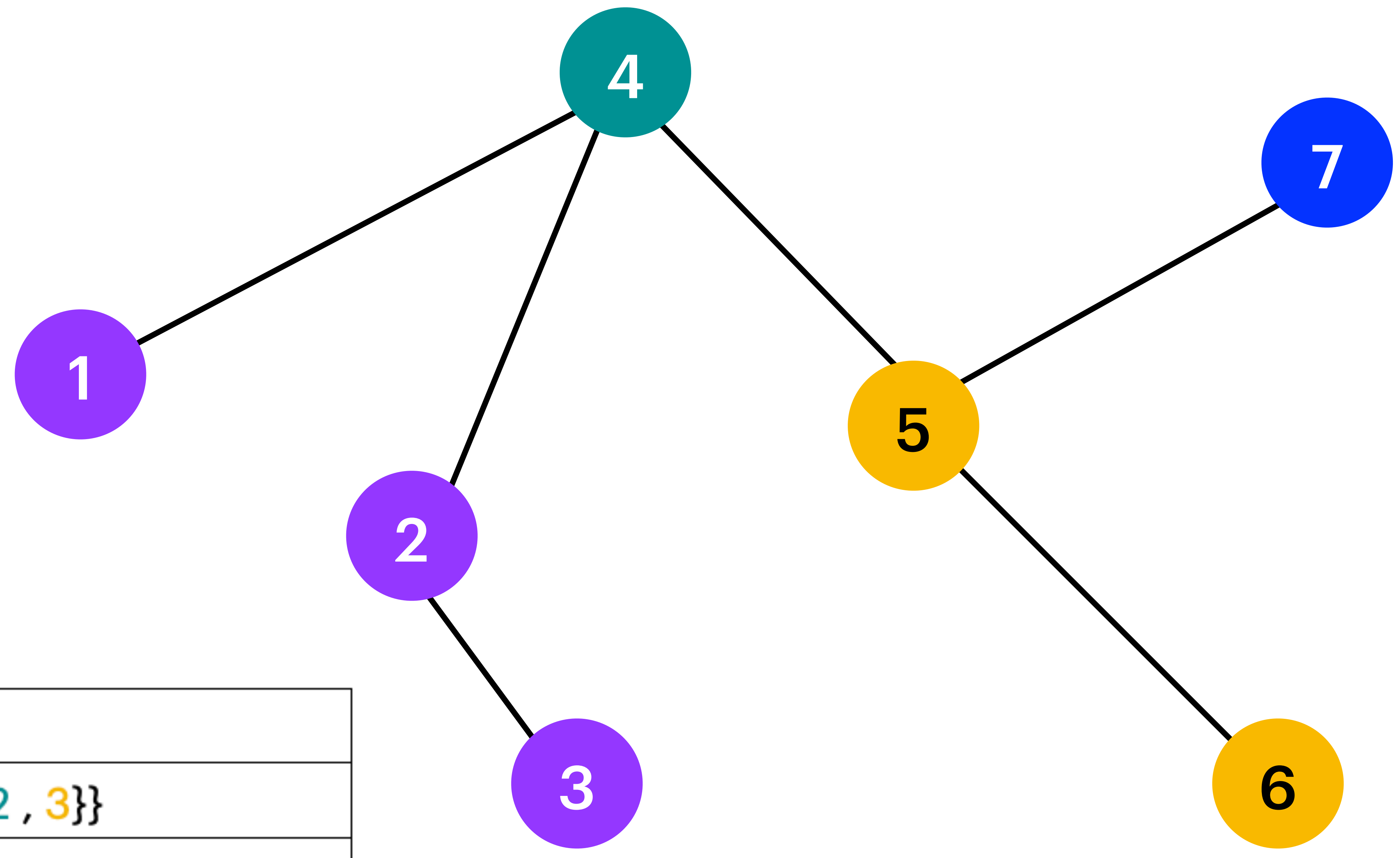
Algorithm 2: $\textsc{Rainbow}(G, \gamma)$

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  $\textsc{Colorful}(G, i)$;
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: $\textsc{Colorful}(G, i)$

forall $v \in V$ **do**
  $P_i(v) \leftarrow \emptyset$;
  forall $x \in N(v)$ **do**
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	
$P_3(2)$	
$P_3(3)$	
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths

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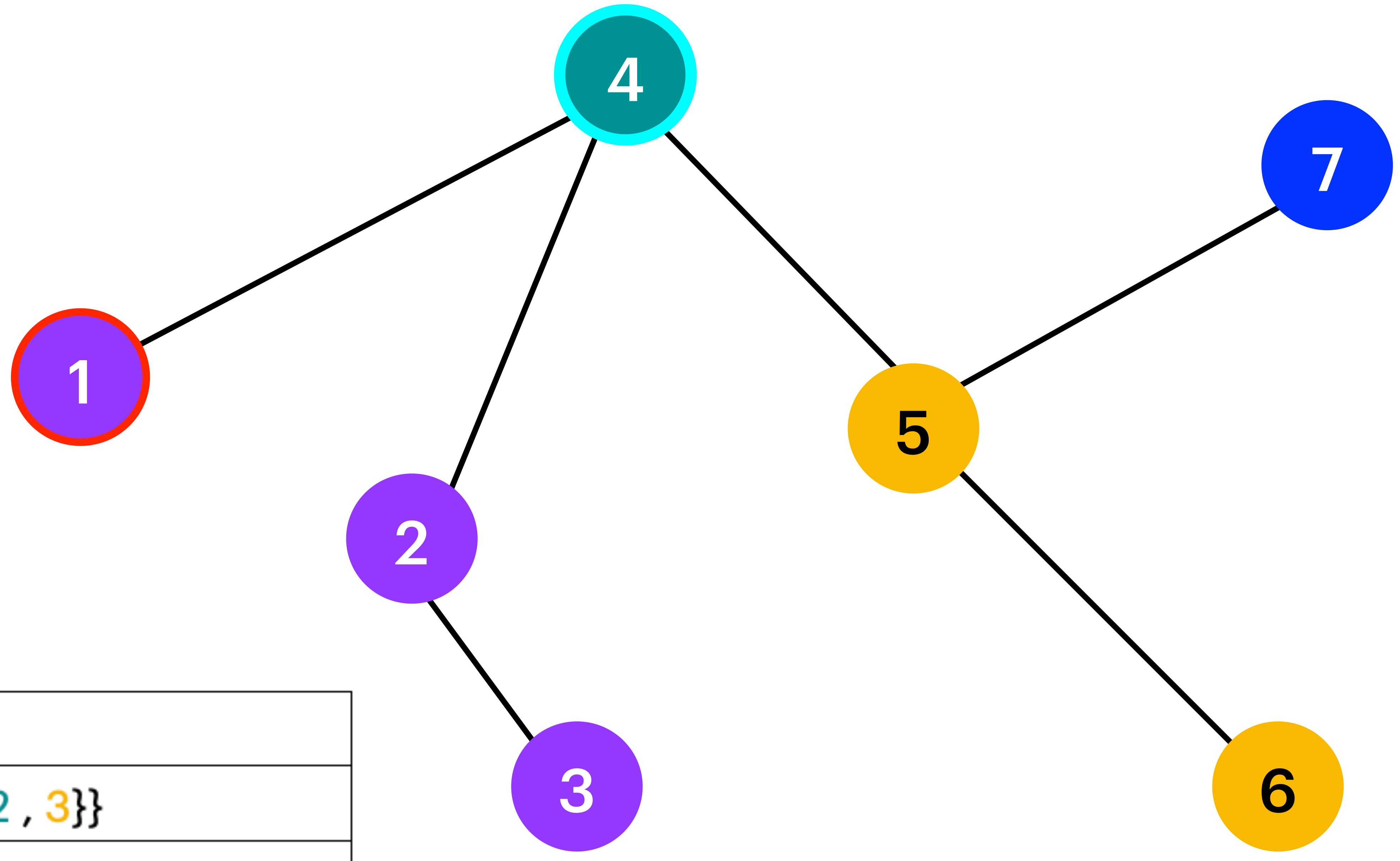
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	
$P_3(2)$	
$P_3(3)$	
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

γ 1 , 2 , 3 , 4

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Colorful Paths

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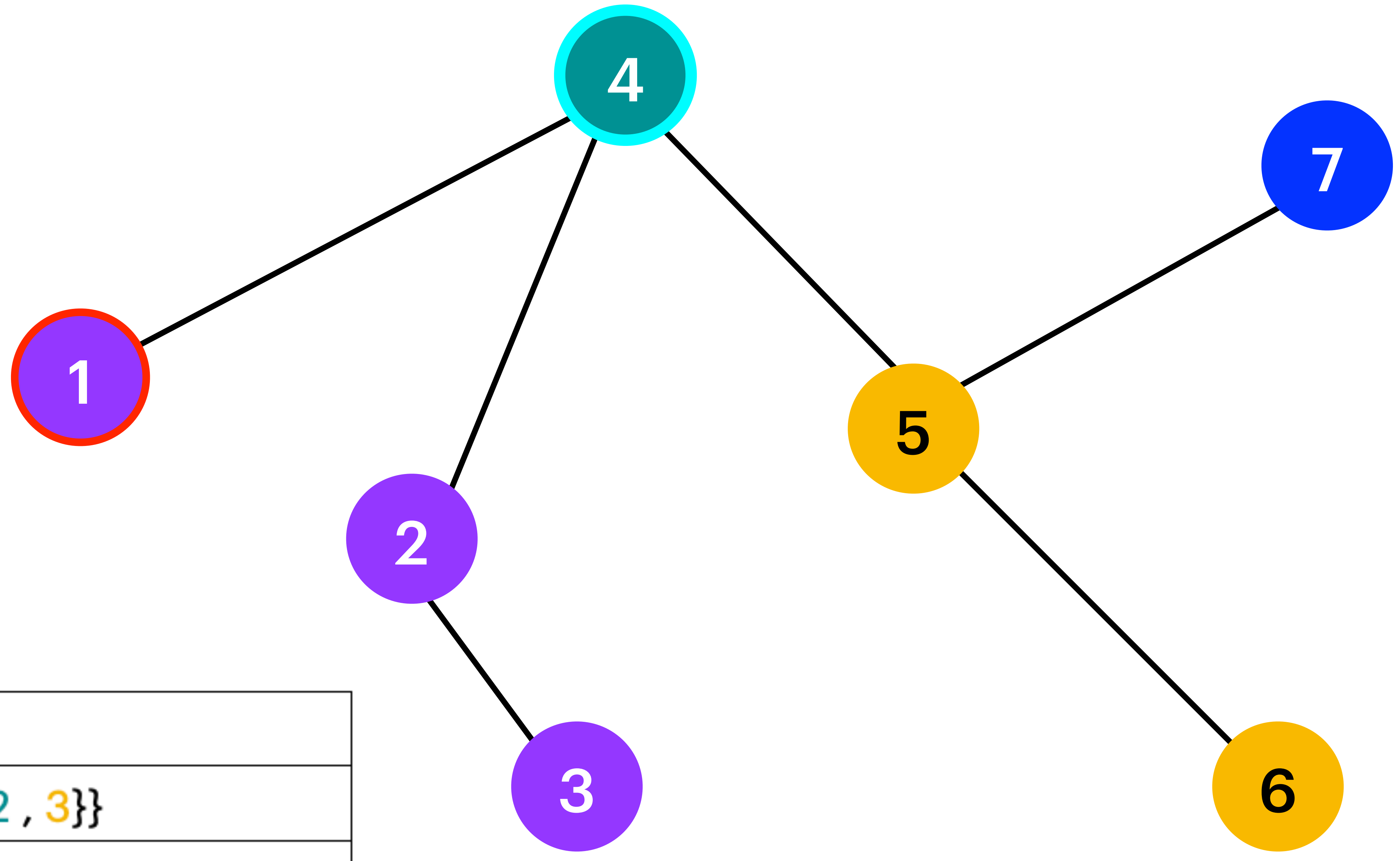
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	
$P_3(3)$	
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths
Algorithm

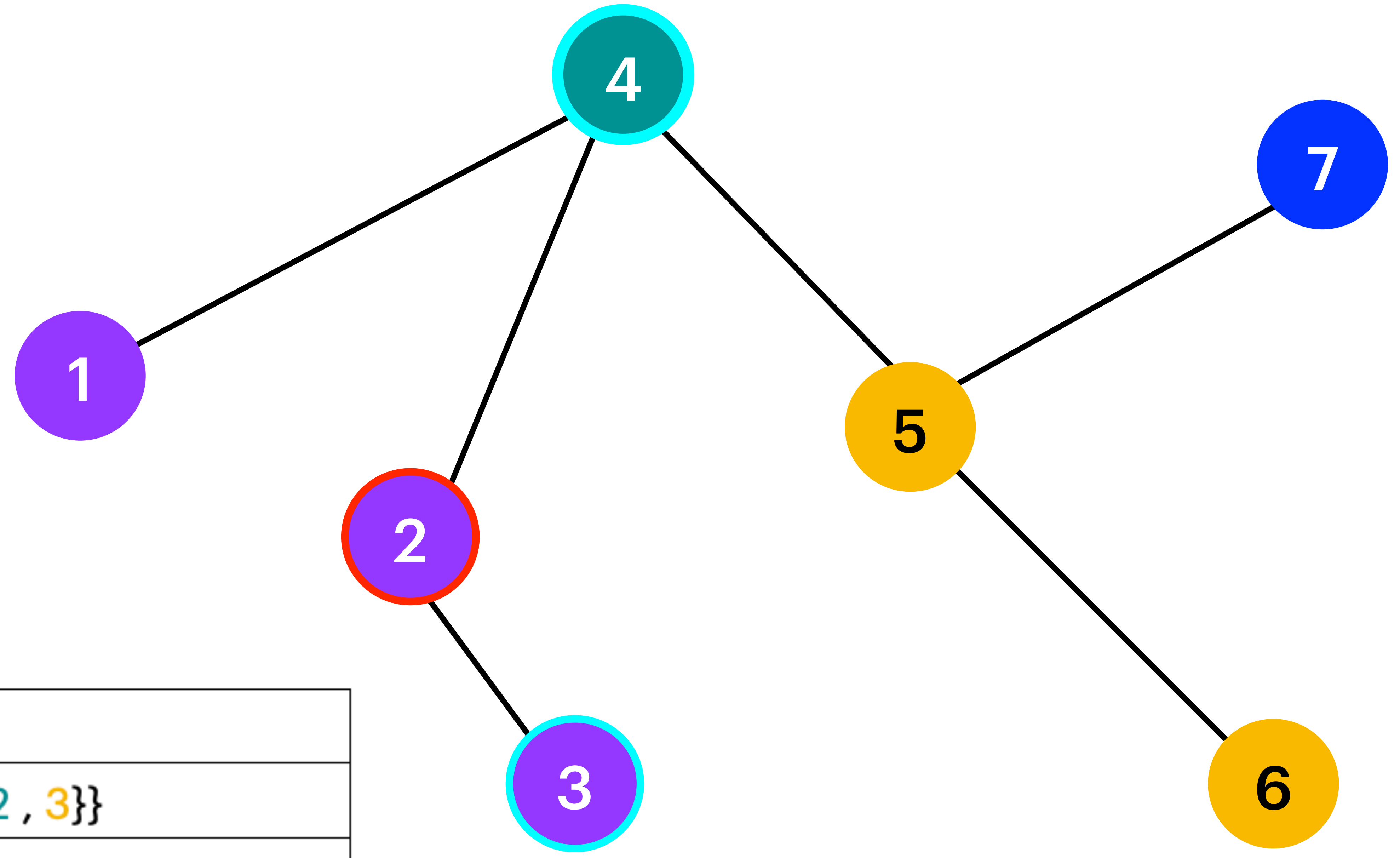
Algorithm 2: RAINBOW(G, γ)

1. **forall** $v \in V$ **do**
2. $\quad P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
3. **for** $i = 1$ **to** $k-1$ **do**
4. $\quad$ COLORFUL(G, i);
5. **return** $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: COLORFUL(G, i)

1. **forall** $v \in V$ **do**
2. $\quad P_i(v) \leftarrow \emptyset$;
3. $\quad$ **forall** $x \in N(v)$ **do**
4. $\quad\quad$ **forall** $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
5. $\quad\quad\quad P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	{{1, 2, 3, 4}}
$P_3(2)$	
$P_3(3)$	
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1, 2, 3}}
$P_2(2)$	{{1, 2, 3}}
$P_2(3)$	∅
$P_2(4)$	{{2, 3, 4}}
$P_2(5)$	{{1, 2, 3}}
$P_2(6)$	∅
$P_2(7)$	{{2, 3, 4}}

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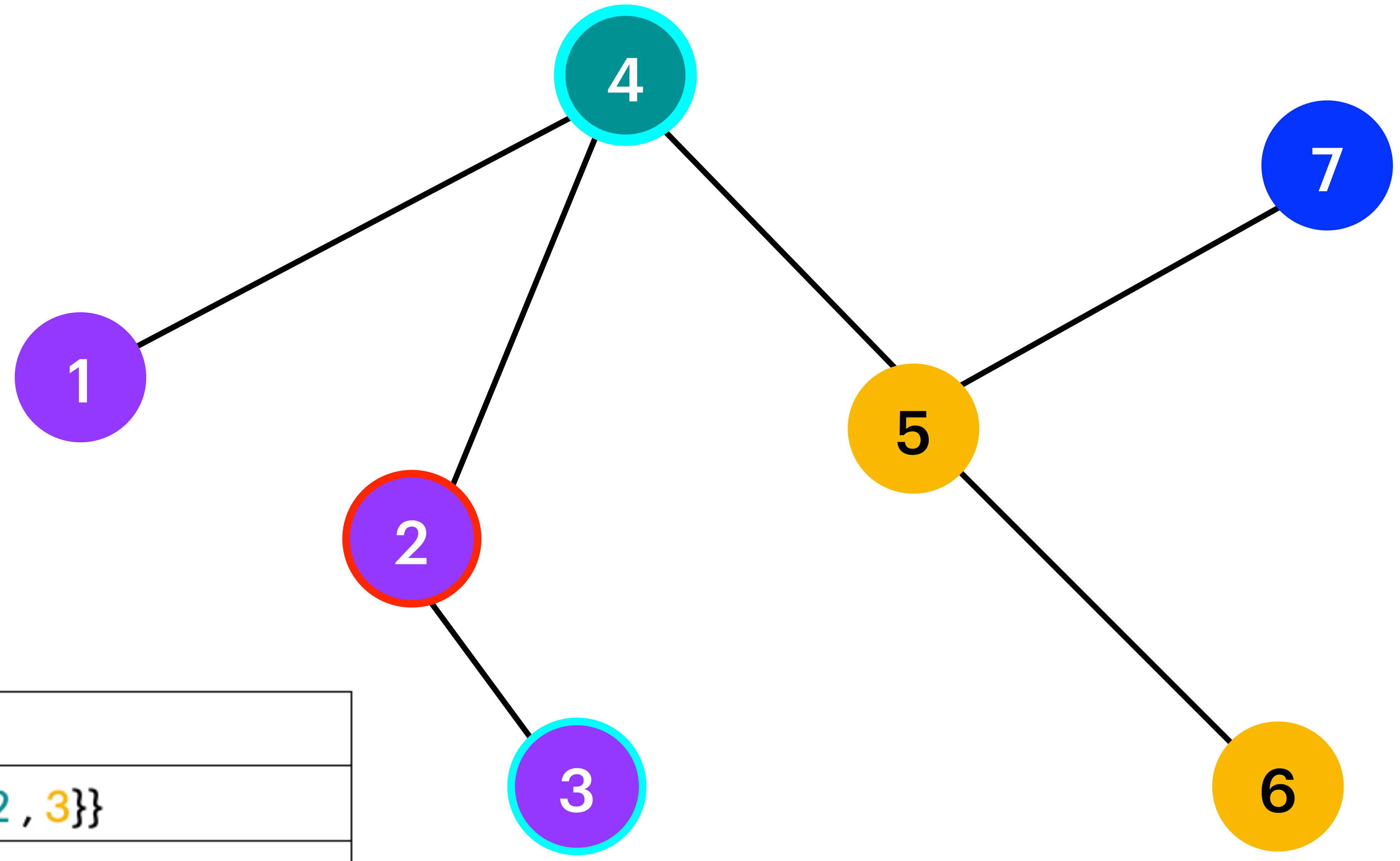
P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
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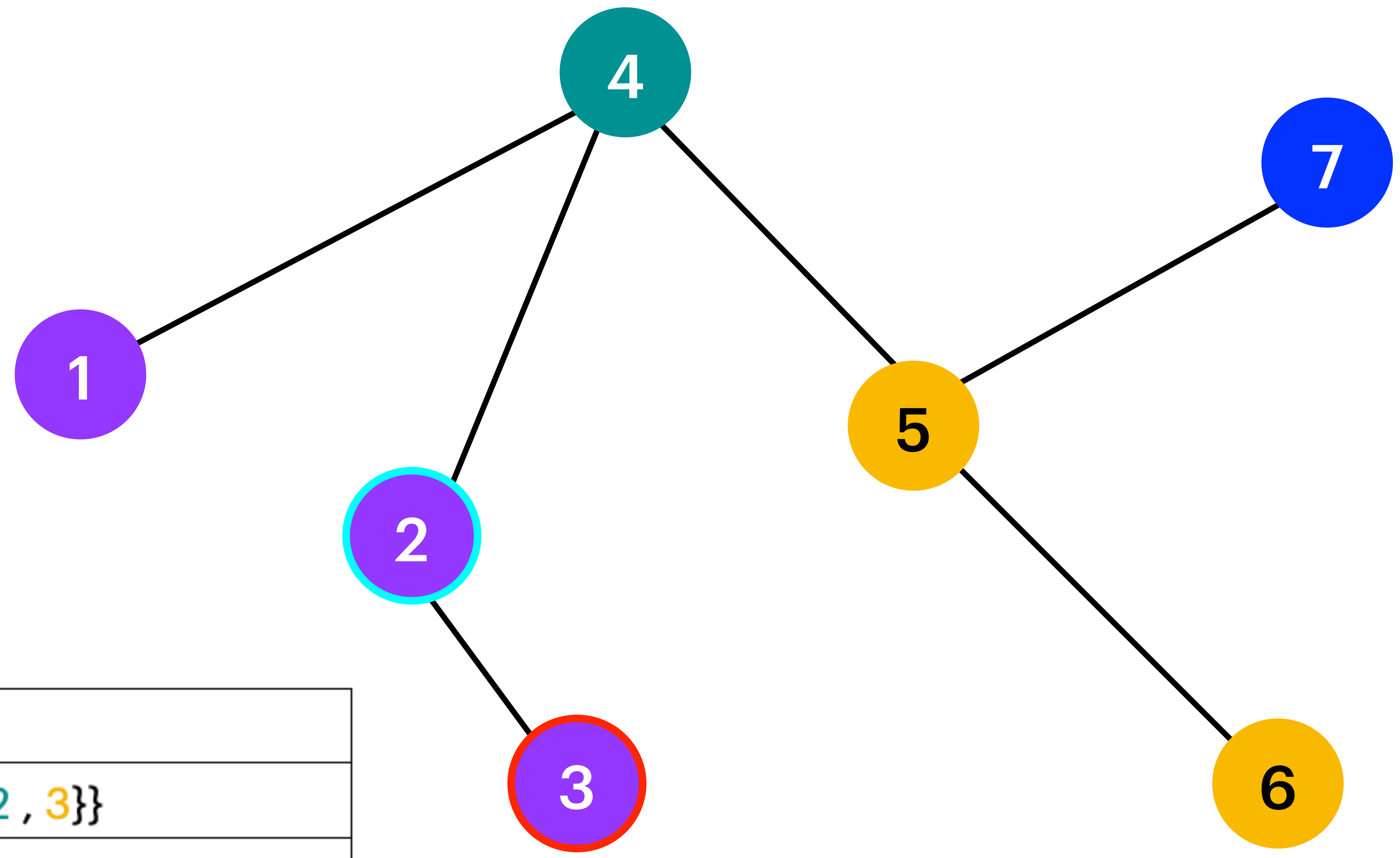
P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
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$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

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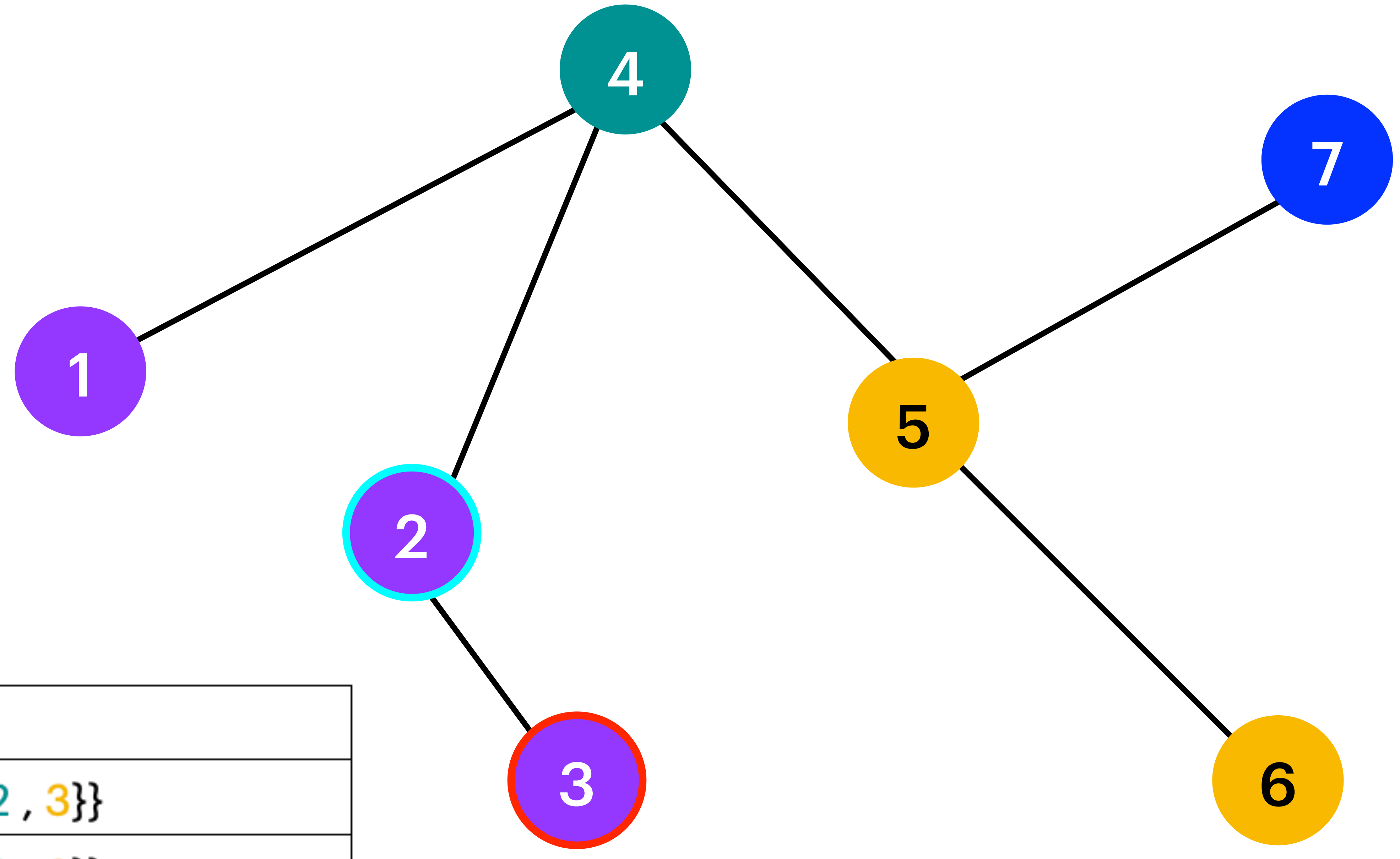
P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	∅
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

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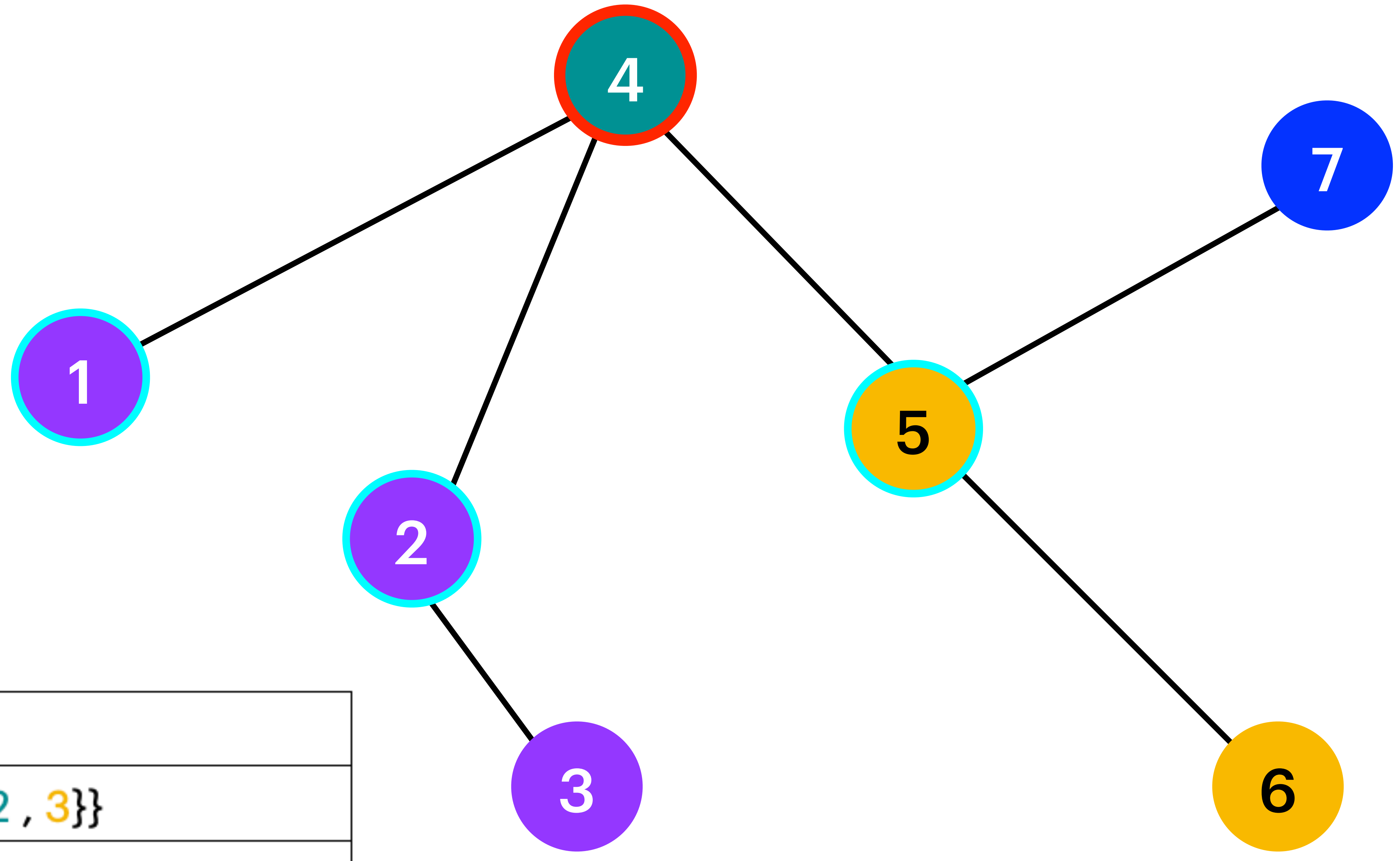
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	∅
$P_3(4)$	
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

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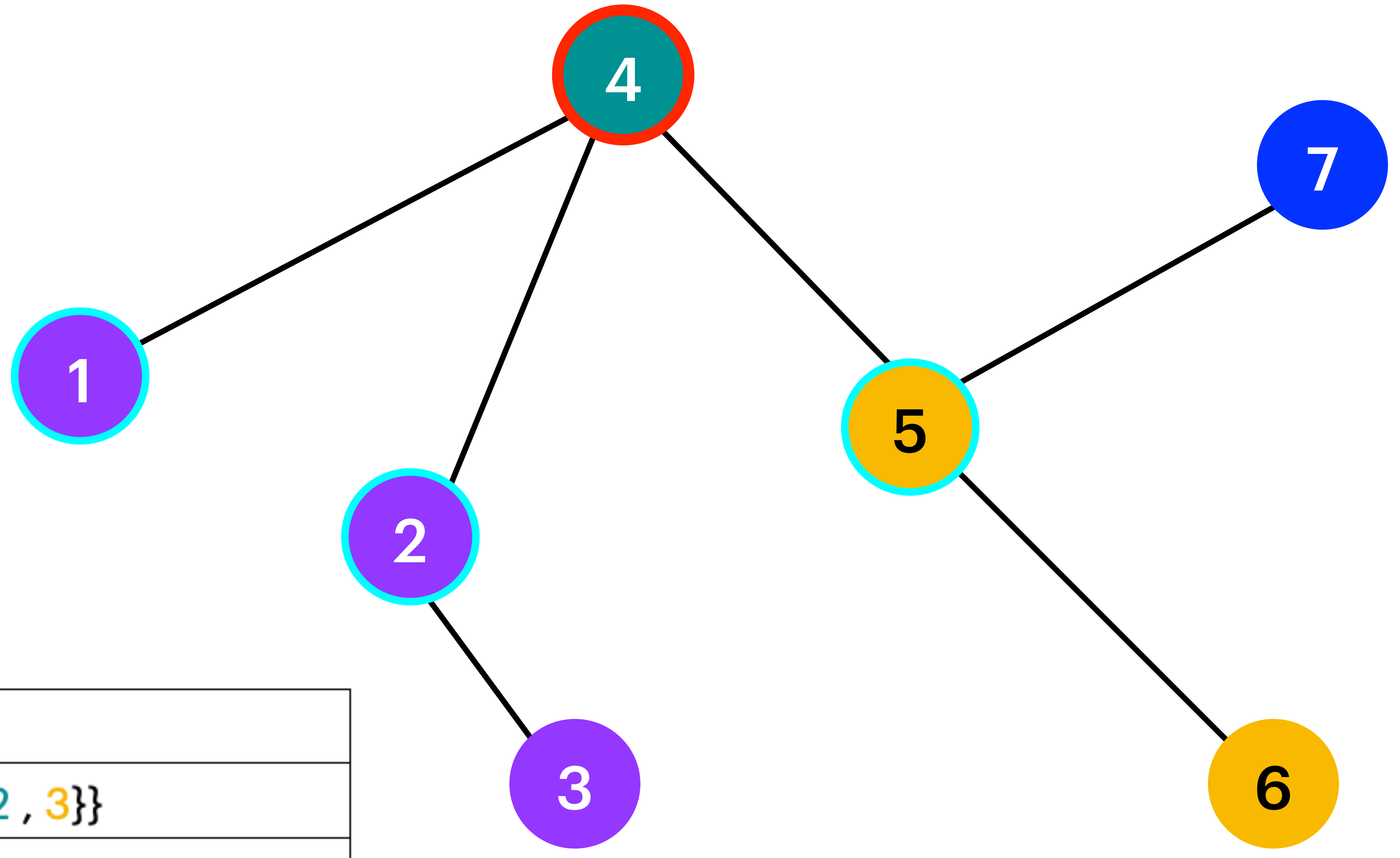
P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	∅
$P_3(4)$	∅
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

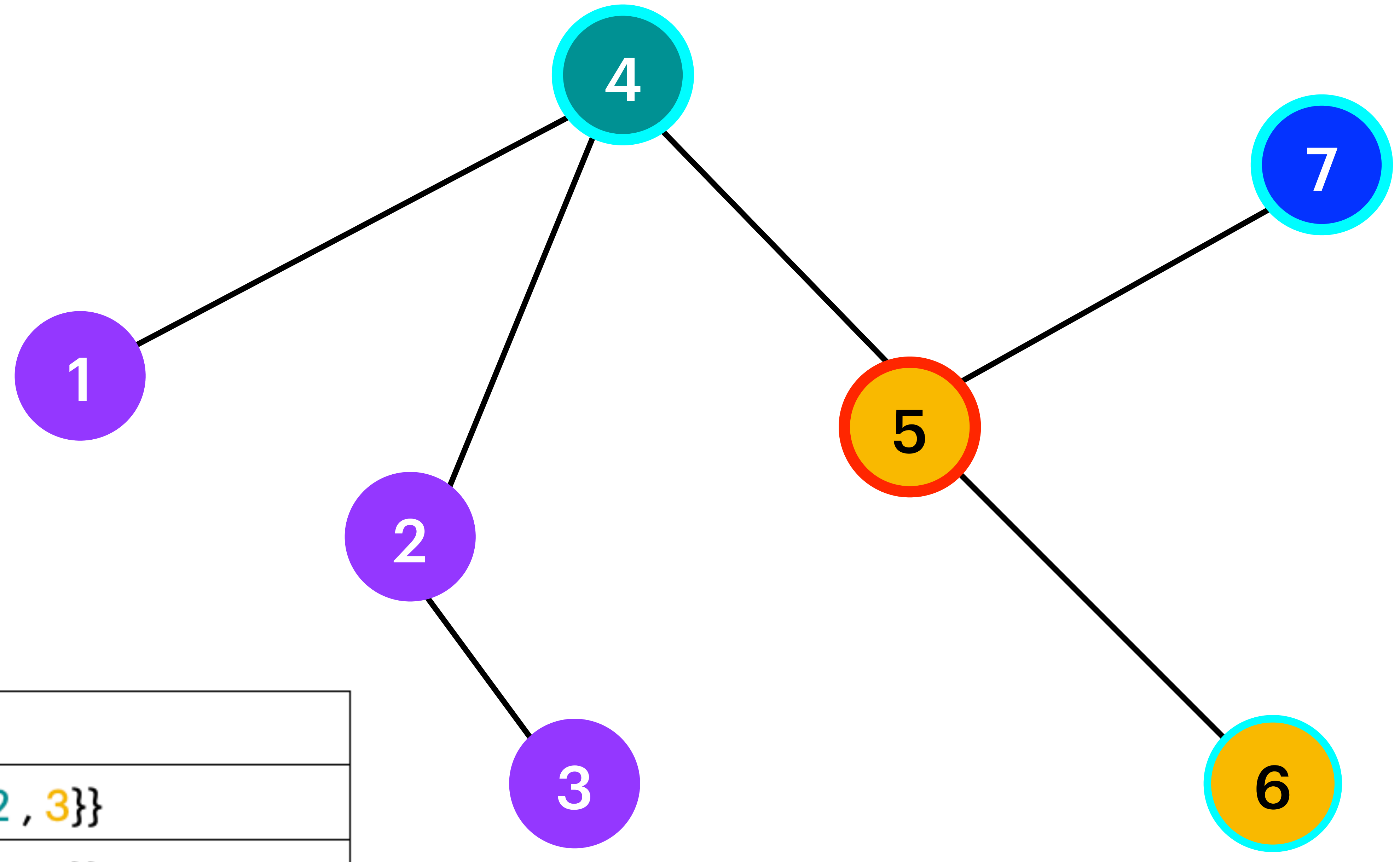
Algorithm 2: $\textsc{Rainbow}(G, \gamma)$

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  $\textsc{Colorful}(G, i)$;
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: $\textsc{Colorful}(G, i)$

forall $v \in V$ **do**
  $P_i(v) \leftarrow \emptyset$;
  forall $x \in N(v)$ **do**
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	∅
$P_3(4)$	∅
$P_3(5)$	
$P_3(6)$	
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

γ 1 , 2 , 3 , 4

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

Colorful Paths

Algorithm

Algorithm 2: RAINBOW(G, γ)

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  COLORFUL(G, i);
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: COLORFUL(G, i)

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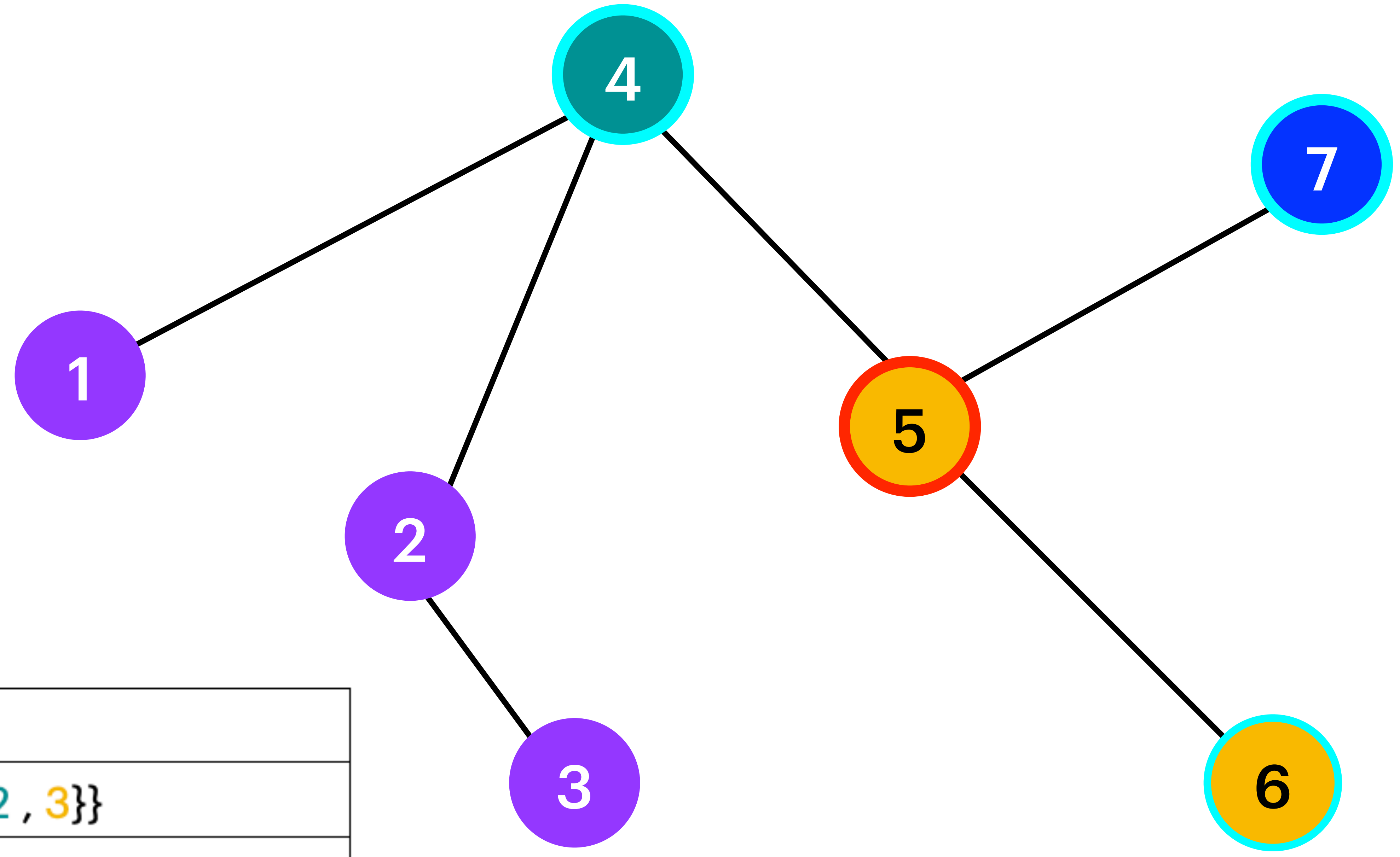
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	∅
$P_3(4)$	∅
$P_3(5)$	∅
$P_3(6)$	
$P_3(7)$	



P_2	
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$P_2(3)$	∅
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$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

γ 1 , 2 , 3 , 4

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

Colorful Paths

Algorithm

Algorithm 2: RAINBOW(G, γ)

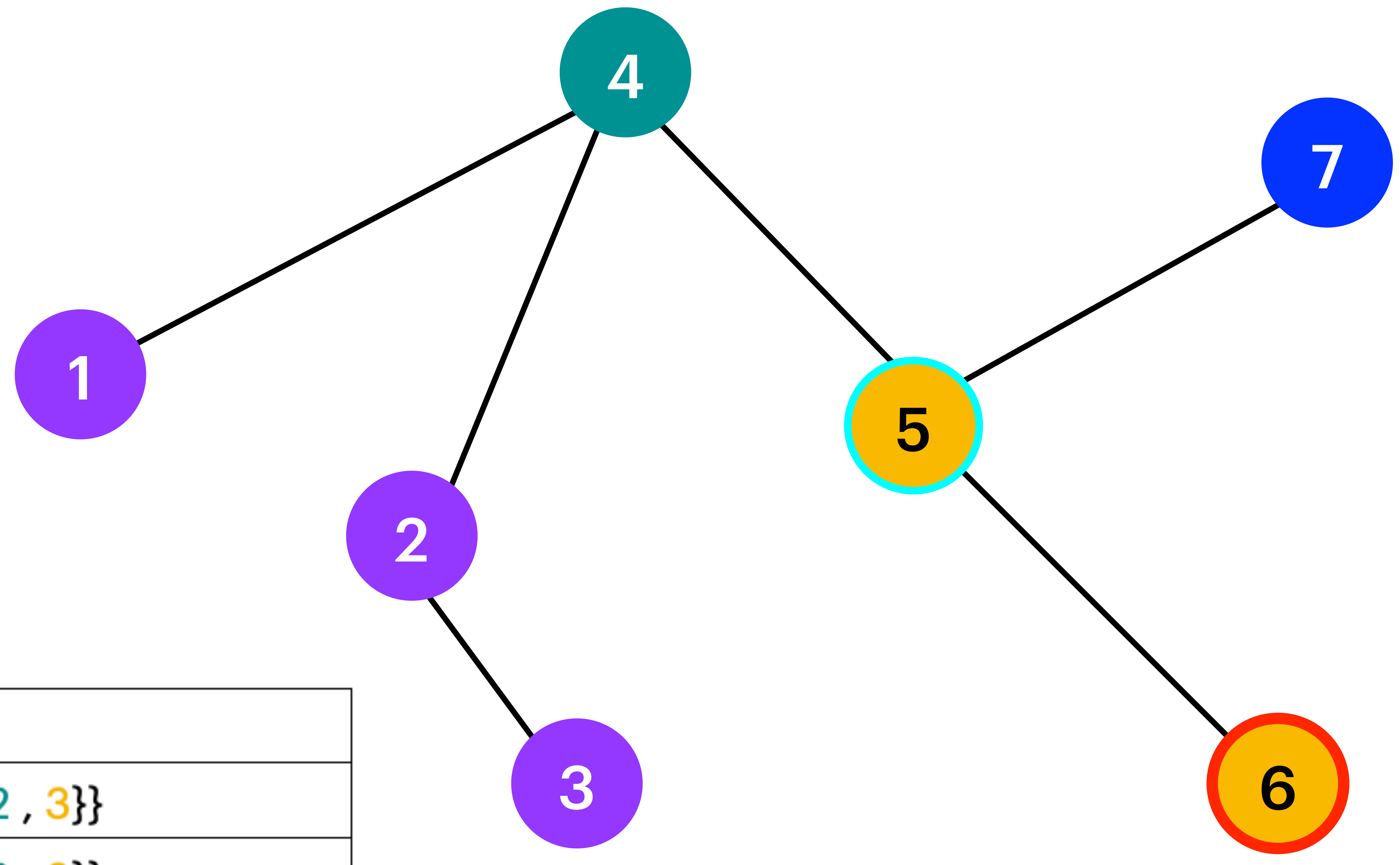
```
forall v ∈ V do
    P_0(v) ← {{γ(v)}};
for i = 1 to k − 1 do
    COLORFUL(G, i);
return ⋃_{v∈V} P_{k−1}(v) ≠ ∅;
```

Algorithm 1: COLORFUL(G, i)

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i−1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

P_3	
$P_3(1)$	$\{\{1, 2, 3, 4\}\}$
$P_3(2)$	$\{\{1, 2, 3, 4\}\}$
$P_3(3)$	$\varnothing$
$P_3(4)$	$\varnothing$
$P_3(5)$	$\varnothing$
$P_3(6)$	
$P_3(7)$	

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S



P_2	
$P_2(1)$	$\{\{1, 2, 3\}\}$
$P_2(2)$	$\{\{1, 2, 3\}\}$
$P_2(3)$	$\varnothing$
$P_2(4)$	$\{\{2, 3, 4\}\}$
$P_2(5)$	$\{\{1, 2, 3\}\}$
$P_2(6)$	$\varnothing$
$P_2(7)$	$\{\{2, 3, 4\}\}$

γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

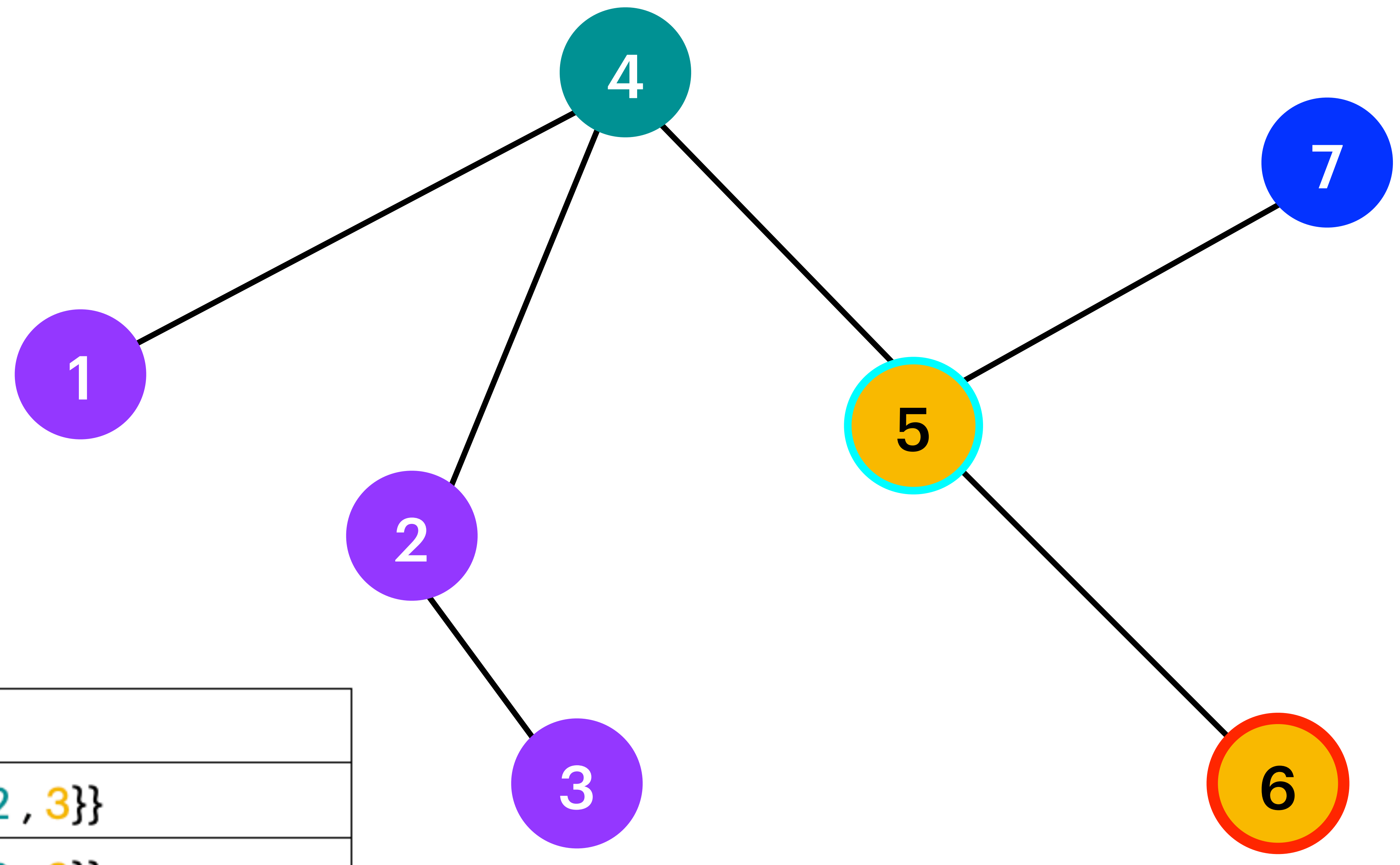
Algorithm 2: RAINBOW(G, γ)

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  COLORFUL(G, i);
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: COLORFUL(G, i)

forall $v \in V$ **do**
  $P_i(v) \leftarrow \emptyset$;
  forall $x \in N(v)$ **do**
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	{{1 , 2 , 3 , 4}}
$P_3(2)$	{{1 , 2 , 3 , 4}}
$P_3(3)$	∅
$P_3(4)$	∅
$P_3(5)$	∅
$P_3(6)$	∅
$P_3(7)$	



P_2	
$P_2(1)$	{{1 , 2 , 3}}
$P_2(2)$	{{1 , 2 , 3}}
$P_2(3)$	∅
$P_2(4)$	{{2 , 3 , 4}}
$P_2(5)$	{{1 , 2 , 3}}
$P_2(6)$	∅
$P_2(7)$	{{2 , 3 , 4}}

γ 1 , 2 , 3 , 4

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

Colorful Paths
Algorithm

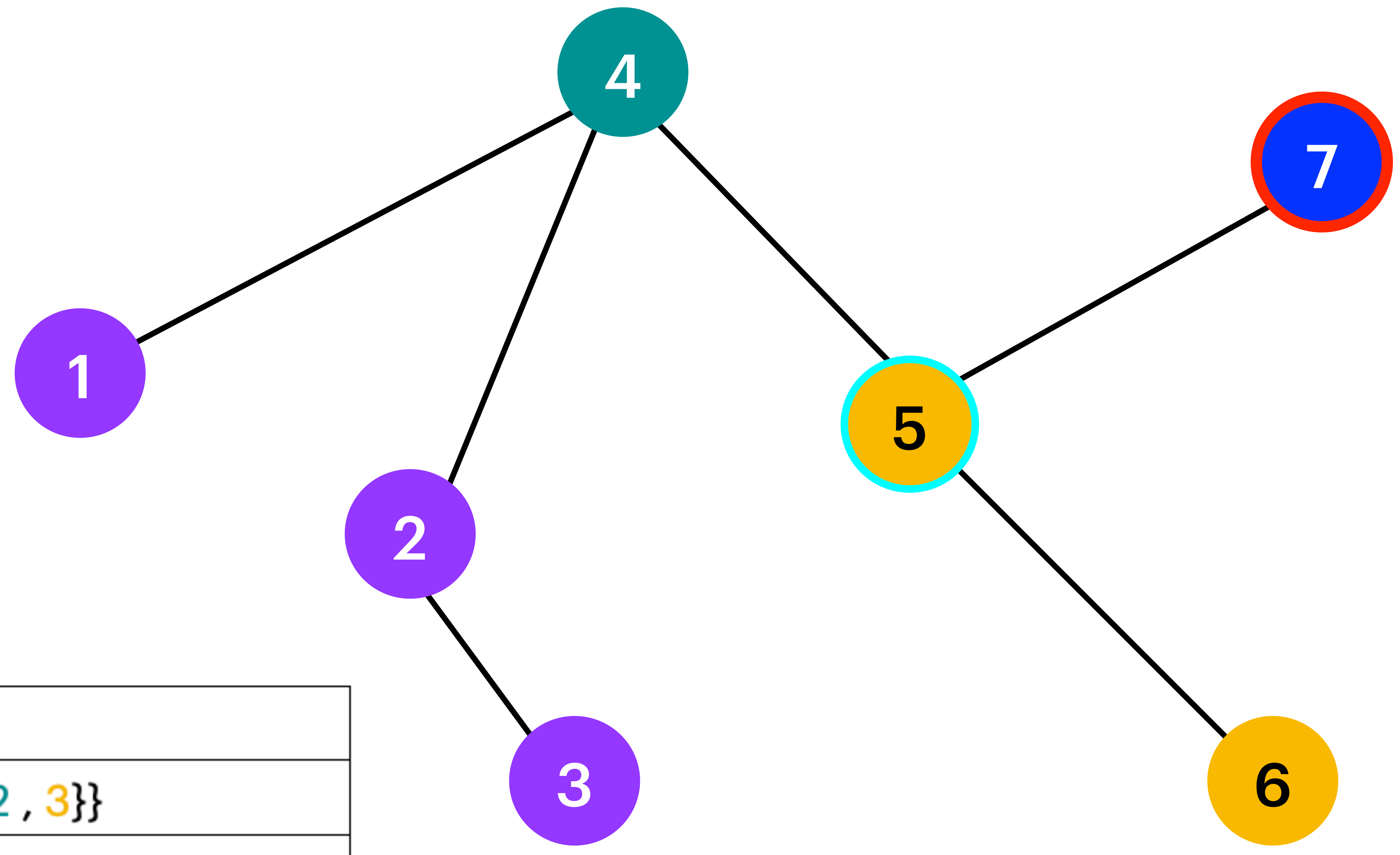
Algorithm 2: $\textsc{Rainbow}(G, \gamma)$

```
forall v ∈ V do
    P_0(v) ← {{γ(v)}};
for i = 1 to k − 1 do
    COLORFUL(G, i);
return ⋃_{v∈V} P_{k−1}(v) ≠ ∅;
```

Algorithm 1: $\textsc{Colorful}(G, i)$

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i−1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

P_3	
P_3 (1)	{{1 , 2 , 3 , 4}}
P_3 (2)	{{1 , 2 , 3 , 4}}
P_3 (3)	∅
P_3 (4)	∅
P_3 (5)	∅
P_3 (6)	∅
P_3 (7)	



P_2	
P_2 (1)	{{1 , 2 , 3}}
P_2 (2)	{{1 , 2 , 3}}
P_2 (3)	∅
P_2 (4)	{{2 , 3 , 4}}
P_2 (5)	{{1 , 2 , 3}}
P_2 (6)	∅
P_2 (7)	{{2 , 3 , 4}}

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

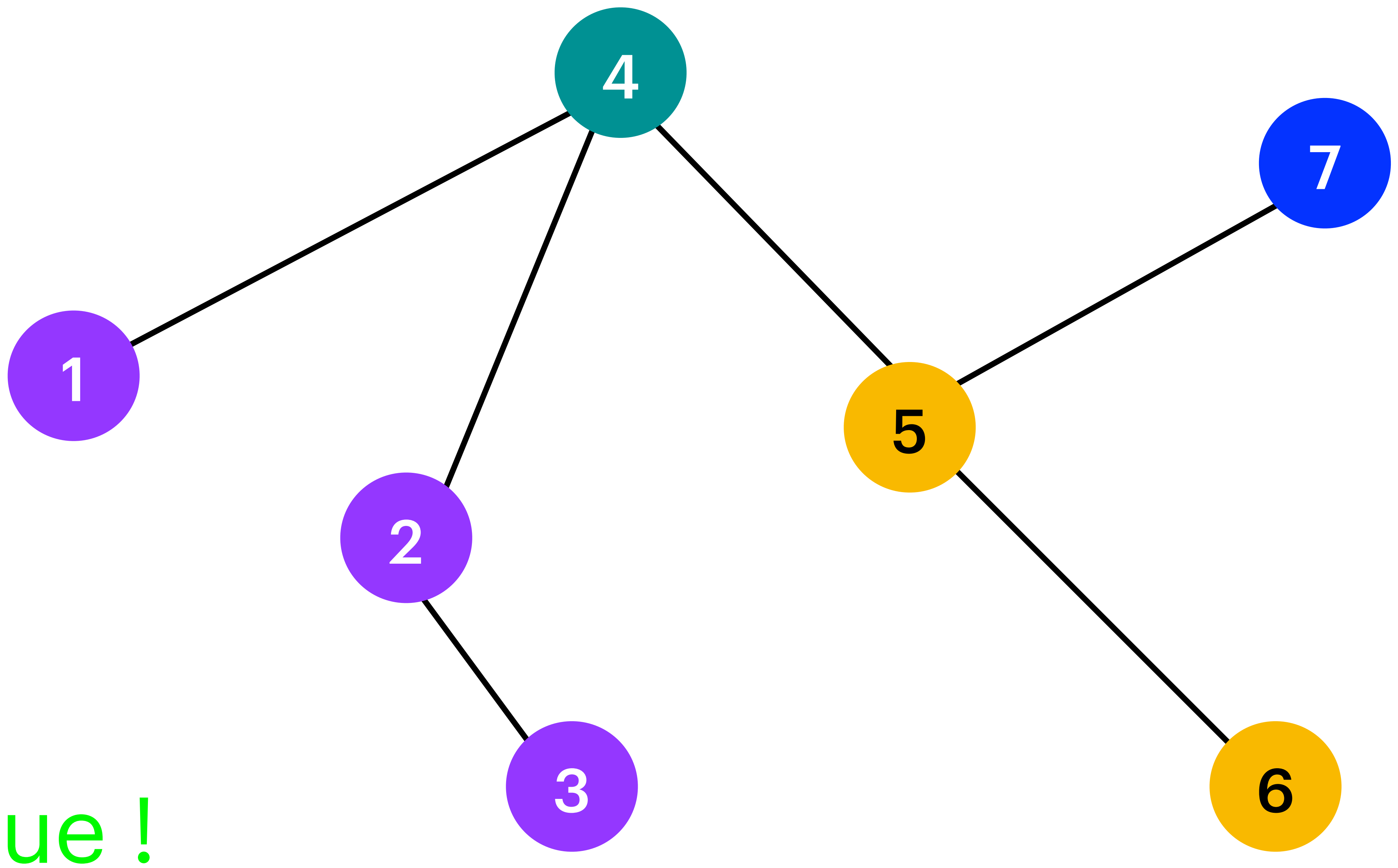
Algorithm 2: RAINBOW(G, γ)

forall $v \in V$ **do**
  $P_0(v) \leftarrow \{\{\gamma(v)\}\}$;
for $i = 1$ **to** $k - 1$ **do**
  COLORFUL(G, i);
return $\bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$;

Algorithm 1: COLORFUL(G, i)

forall $v \in V$ **do**
  $P_i(v) \leftarrow \emptyset$;
  forall $x \in N(v)$ **do**
    forall $R \in P_{i-1}(x)$ *such that* $\gamma(v) \notin R$ **do**
      $P_i(v) \leftarrow P_i(v) \cup \{R \cup \{\gamma(v)\}\}$;

P_3	
$P_3(1)$	$\{\{1, 2, 3, 4\}\}$
$P_3(2)$	$\{\{1, 2, 3, 4\}\}$
$P_3(3)$	$\varnothing$
$P_3(4)$	$\varnothing$
$P_3(5)$	$\varnothing$
$P_3(6)$	$\varnothing$
$P_3(7)$	$\{\{1, 2, 3, 4\}\}$



returns true !

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

γ 1 , 2 , 3 , 4

Colorful Paths

Algorithm

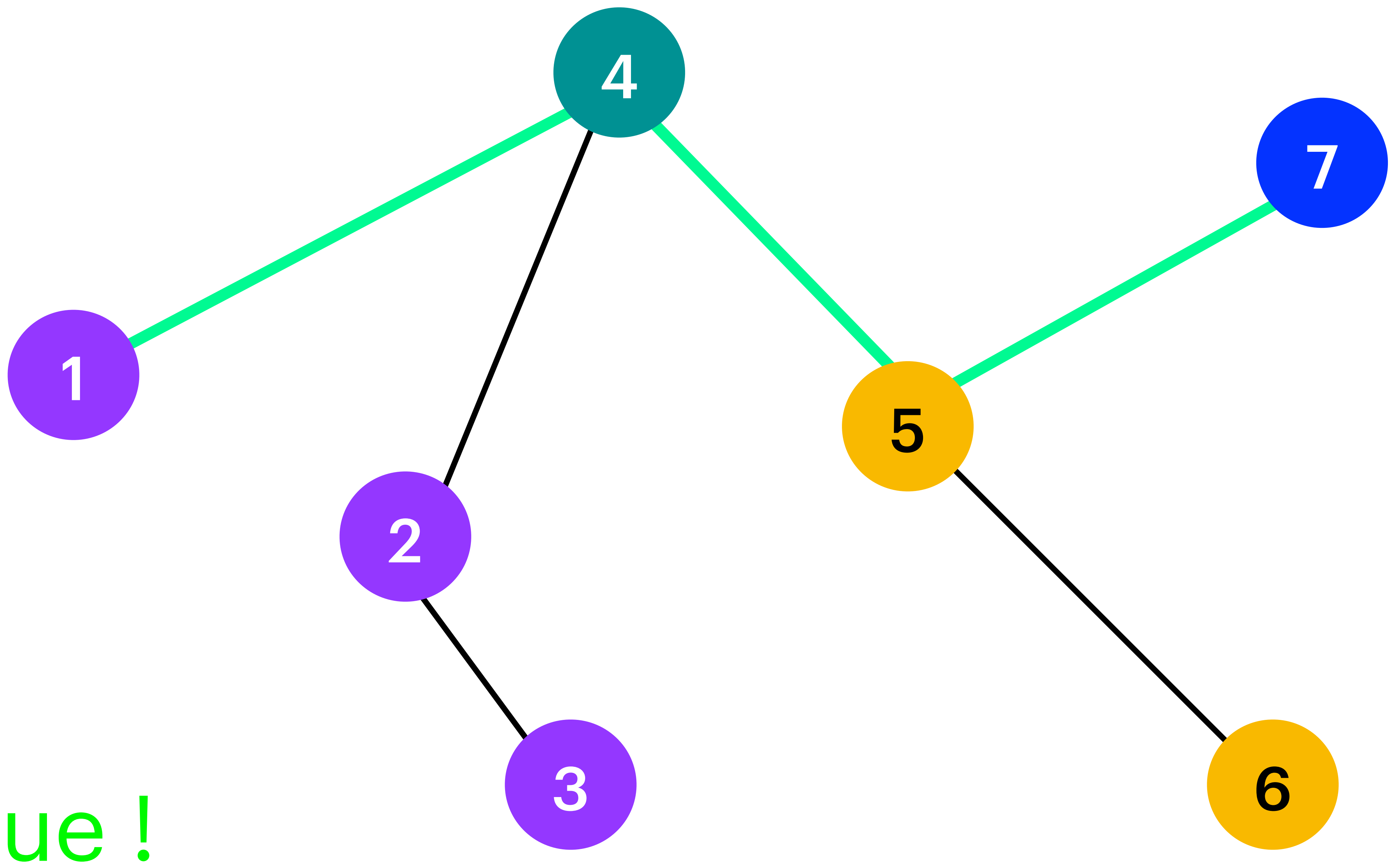
Algorithm 2: RAINBOW(G, γ)

```
forall v ∈ V do
    P_0(v) ← {{γ(v)}};
for i = 1 to k − 1 do
    COLORFUL(G, i);
return ⋃_{v∈V} P_{k−1}(v) ≠ ∅;
```

Algorithm 1: COLORFUL(G, i)

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i−1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

P3	
P3 (1)	{{1 , 2 , 3 , 4}}
P3 (2)	{{1 , 2 , 3 , 4}}
P3 (3)	∅
P3 (4)	∅
P3 (5)	∅
P3 (6)	∅
P3 (7)	{{1 , 2 , 3 , 4}}



returns true !

color sets S s.t |S| = i+1 and there is a colorful path of length i ending at v only using the colors in S

given : A graph $G = (V, E)$

A coloring of its vertices with k colors $\gamma : V \to [k]$

to find : Does there exist a colorful path of length $k - 1$ in a randomly colored graph ? A path is colorful if all of the vertices in the path have a different color

∃ colorful path of length $k-1 \iff \bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$

1 , 2 , 3 , 4

Colorful Paths

Algorithm + Probability

given : A graph $G = (V, E)$

A coloring of its vertices with k colors $\gamma : V \to [k]$

to find : Does there exist a colorful path of length $k - 1$ in a randomly colored graph ? A path is colorful if all of the vertices in the path have a different color

$\exists$ colorful path of length $k-1 \iff \bigcup_{v \in V} P_{k-1}(v) \neq \emptyset$

$$\mathcal{O}(2^k \cdot k \cdot m)$$

Algorithm 1: COLORFUL(G, i) — G a γ-colored graph

```
forall v ∈ V do
    P_i(v) ← ∅;
    forall x ∈ N(v) do
        forall R ∈ P_{i-1}(x) such that γ(v) ∉ R do
            P_i(v) ← P_i(v) ∪ {R ∪ {γ(v)}};
```

Algorithm 2: RAINBOW(G, γ) — G a graph, γ a k-coloring

```
forall v ∈ V do
    P_0(v) ← {{γ(v)}};
for i = 1 to k − 1 do
    COLORFUL(G, i);
return ⋃_{v∈V} P_{k−1}(v) ≠ ∅;
```

Satz 3.2

Let G be a graph that contains a path of length $k - 1$.

1. A random coloring of the graph using k colors produces a colorful path of length $k - 1$ with probability at least

$$p_{\text{success}} \geq \frac{k!}{k^k} \geq e^{-k}$$

2. If we repeat the coloring process multiple times, then the **expected number of repetitions** needed until success is at most

$$\frac{1}{p_{\text{success}}} \leq e^k$$

Satz 3.3

1. The full randomized algorithm (including repetitions) has a runtime of

$$\mathcal{O}(\lambda \cdot (2e)^k \cdot km)$$

where $\lambda > 1$ is a tunable parameter (confidence level).

2. If the algorithm answers **YES**, then the graph **does contain** a path of length $k - 1$.
3. If the graph does contain a path of length $k - 1$, then the probability that the algorithm answers **NO** is at most

$$e^{-\lambda}$$

Questions

Feedbacks , Recommendations

Nil Ozer

Helper

Mathematical Tools and Notations

$[n] := \{1, 2, \ldots, n\}$

$[n]^k :=$ the set of sequences over $[n]$ of length k $\qquad |[n]^k| = n^k$

$\binom{[n]}{k} :=$ the set of k-element subsets of $[n]$ $\qquad \left|\binom{[n]}{k}\right| = \binom{n}{k}.$

The k nodes on a path of length $k-1$ can be colored using $[k]$ in exactly k^k ways

$k!$ of these colorings use each color exactly once

Helper

Mathematical Tools and Notations

Handshaking lemma : For all graphs , it holds that $\sum_{v \in V} \deg(v) = 2|E|$.

If you repeat an experiment with success probability p until success, then the expected number of trials is $\frac{1}{p}$ ($Geo(p)$)

Helper

Mathematical Tools and Notations

For $c, n \in \mathbb{R}^+$, it holds that $c^{\log n} = n^{\log c}$

$$2^{\log n} = n^{\log 2} = n \text{ and } 2^{\mathcal{O}(\log n)} = n^{\mathcal{O}(1)} \text{ is always polynomial in } n$$

For $n \in \mathbb{N}_0$, it holds that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ (binomial theorem)

For $n \in \mathbb{N}_0$, it holds that $\frac{n!}{n^n} \geq e^{-n}$ (power series expansion of the exponential function)