

Mini-exam (Induction + Asymptotic Notation)

/ 5 P

a) Asymptotic notation quiz: For each of the following claims, state whether it is true or false. You get 1P for a correct answer, -1P for a wrong answer, 0P for a missing answer. You get at least 0 points in total.

FS23

Assume $n \geq 4$.

Claim	true	false
$\lim_{n \rightarrow \infty} \frac{n^3 + n^4}{n^4} = \lim_{n \rightarrow \infty} \frac{n^4(\frac{1}{n} + 1)}{n^4} = 1$ $n^3 + n^4 = \Theta(n^4)$	☒	☐
$n^x > \log(n)^{100}$ $\lim_{n \rightarrow \infty} \frac{n^{10}}{\log(n)^{100}} = \frac{10n^9}{100 \log(n)} \xrightarrow{\text{L'Hopital}} \frac{1}{n} \rightarrow 0$ $n^{10} \leq O(\log(n)^{100})$	☐	☒
$n! > \frac{n}{2} \cdot 2^n > 2^n$ $1 \cdot 2 \cdot 3 \cdot \dots \cdot n \leq O(2^n)$	☐	☒
Suppose $a_1 = 1$ and $a_{i+1} = 3a_i + 1$ for all $i \geq 1$. Then $a_n \leq O(4^n)$.	☒	☐
$2^{10 \log n} = 2^{10 \log n - 20 \log n} = 2^{-10 \log n} = \frac{1}{2^{10 \log n}} \rightarrow 0$ $2^{10 \log(n)} = \Theta(2^{20 \log(n)}) \leq O(2^{\log(n)})$	☐	☒

$a_1 = 1 = 1$
 $a_2 = 4 = 3^1 + 3^0$
 $a_3 = 13 = 3^2 + 3^1 + 3^0$
 $a_4 = 40 = 3^3 + 3^2 + 3^1 + 3^0$
↳ geometric series
 $a_n = \sum_{k=0}^{n-1} 3^k = \frac{1-3^n}{1-3} = \Theta(3^n) \Rightarrow \leq O(4^n)$

/ 4 P

b) Induction: Consider the Fibonacci numbers $(F_n)_{n \in \mathbb{N}}$, which are given by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Show by mathematical induction that for any integer $n \geq 0$,

HS21

$$F_{n+1}^2 \geq \sum_{k=0}^n F_k^2.$$

Hint: Use the facts that $F_{n+1} \geq F_n$ and $F_n \geq 0$ for all $n \in \mathbb{N}$ (you don't need to justify that).

Base Case: $n=0$

$$F_1^2 = 1 \geq \sum_{k=0}^0 F_k^2 = F_0^2 = 0$$

$n=1$

$$F_2^2 \stackrel{\text{def}}{=} 1 \geq \sum_{k=0}^1 F_k^2 = F_0^2 + F_1^2 = 0 + 1 = 1$$

I.H.: For some $m \geq 1$

$$F_{m+1}^2 \geq \sum_{k=0}^m F_k^2$$

I.S.: $m \rightarrow m+1$:

$$\begin{aligned} F_{m+1+1}^2 &\stackrel{\text{def}}{=} (F_{m+1} + F_m)^2 \\ &= F_{m+1}^2 + 2F_{m+1}F_m + F_m^2 \\ &\geq \sum_{k=0}^m F_k^2 + 2F_{m+1}F_m + F_m^2 \\ &\geq \sum_{k=0}^m F_k^2 + 2F_{m+1}F_m \quad | F_m \geq 0, F_m^2 \geq 0 \\ &= \sum_{k=0}^m F_k^2 + F_{m+1}(F_m + F_m) \\ &\geq \sum_{k=0}^m F_k^2 + \underbrace{F_{m+1}(F_m + F_{m-1})}_{= F_{m+1}^2} \quad | F_m \geq F_{m-1} \\ &\geq \sum_{k=0}^{m+1} F_k^2 \end{aligned}$$

(By the principle ...)