

A&D exercise class 28.09.2026

Plan for today:

- Moodle quiz
- Discussion of Moodle quiz
- Input: Asymptotic growth
- In-class exercises
- Discussion of exercise sheet 1
- Peer grading

Moodle quiz:

- We will start with the Moodle quiz.
- Please prepare your device and open the quiz («Moodle Quiz Week 3»; <https://moodle-app2.let.ethz.ch/mod/quiz/view.php?id=1455045>) until you only need to give the password.
- You can choose the language of the quiz by choosing the language on Moodle (on Moodle, click on your initials, then language).
- If you have any problems with the setup, please, let me know!

Moodle quiz

- Password: induction5
- Duration: 15 minutes

Moodle quiz – discussion (pasture break)

Consider the pasture break problem, where the distance k to the gap in the fence is known in advance. There is an algorithm that always finds this gap in at most $3k$ steps.

- ☒ True
- ☐ False

Consider the pasture break problem, where the distance k to the gap in the fence is **not** known in advance. There is an algorithm that always finds this gap in at most $3k$ steps.

- ☐ True
- ☒ False

Consider the pasture break problem, where the distance k to the gap in the fence is **not** known in advance. Then, it is an efficient strategy to

- ☐ ...walk as far to the left as possible until the gap is found.
- ☐ ...walk as far to the right as possible until the gap is found.
- ☒ ...walk in a 'zick-zack' until the gap is found.

Moodle quiz – discussion (star search)

Consider the problem of finding a star p_s among $n \geq 2$ people $p_1, \dots, p_n$. There is an algorithm that finds a star if one exists, using at most $O(n)$ questions (meaning that there is a constant C such that the algorithm uses at most $C \cdot n$ questions).

- ☒ True
- ☐ False

Consider the problem of finding a star p_s among $n \geq 2$ people $p_1, \dots, p_n$. Exactly one of the following statements is always true. Choose the correct one.

- ☐ There always exists no star among these people.
- ☐ There may exist two distinct stars among these people.
- ☒ There always exists either exactly one or no star among these people.
- ☐ There always exists exactly one star among these people.

Consider the problem of finding a star p_s among $n \geq 2$ people $p_1, \dots, p_n$. Assume that there exists a star. With your first question you find out that p_1 knows p_2 . Which of the following strategies will always correctly find the star?

- ☒ Completely disregard the person p_1 , and continue to recursively find a star among $p_2, \dots, p_n$.
- ☐ Completely disregard the person p_2 , and continue to recursively find a star among $p_1, p_3, p_4, \dots, p_n$.
- ☐ Neither of the other options, to find a star we also need to know whether or not p_2 knows p_1 .

Recall: p_s is a star if

- all p_i knows p_s
- p_s knows no p_i

Moodle quiz – discussion (O-notation)

$$n^2 + 2n \leq O(n^2).$$

- ☐ True
☐ False

$$1 + 2 + 3 + \dots + n \leq O(n).$$

- ☐ True
☐ False

Enter the smallest positive integer $k \in \mathbb{N}$ such that $2n + 10n^5 + 2 \cdot (\log n)^3 \leq O(n^k)$.

Answer:

Moodle quiz – discussion (asymptotic growth)

The function $f(n) = 100^n$ ($f : \mathbb{N} \rightarrow \mathbb{R}^+$) grows asymptotically faster than the function $g(n) = 99^n$ ($g : \mathbb{N} \rightarrow \mathbb{R}^+$). Recall that we say that f grows asymptotically faster than g if $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 0$.

☒ True

☐ False

Induction – recap

Goal: Prove some statement $A(n)$ for all $n \in \mathbb{N}$

Base case: Show $A(1)$.

Induction hypothesis: Assume $A(k)$ holds for **some** $k \in \mathbb{N}$.

Induction step: Prove $A(k + 1)$ using $A(k)$.

Moodle quiz – discussion (induction)

Let $A(n)$ be a mathematical statement (for $n \in \mathbb{N}$). Assume that we know the following:

- $A(2)$ is true, and
- for every $n \in \mathbb{N}$, if $A(n)$ is true, then also $A(n+1)$ is true.

Which of the following are then necessarily true? Select all options that are necessarily true.

- ☐ $A(n)$ is true for all $n \in \mathbb{N}$.
- ☐ $A(1)$ is true.
- ☐ $A(3)$ is true.

For all real numbers $x \geq -1$ and positive integers $n \in \mathbb{N}$, the following holds: $(1+x)^n \geq 1+nx$.

- ☐ True
- ☐ False

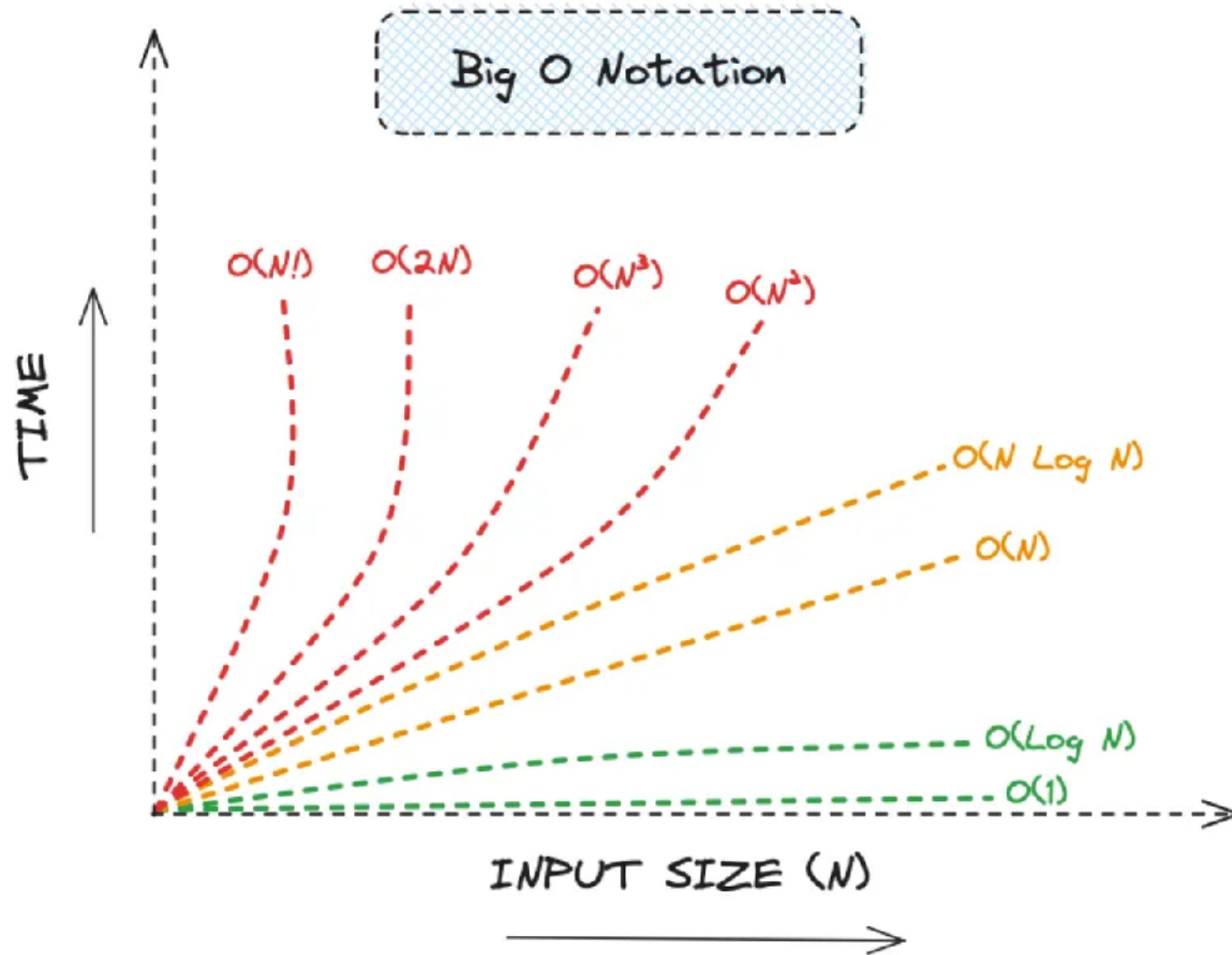
Asymptotic Notation

O-Notation

- Describes the upper-bound of an algorithm
- To express the worst-case running time
- Describes the performance of an algorithm as the amount of data increases
- As the input size of the algorithm grows, the execution time of the algorithm is also going to grow. (Constant, linear, logarithmic, exponential...)
- n represents the amount of data that we're passing in
- We don't care about the differences, when those differences are constant values. We care about n

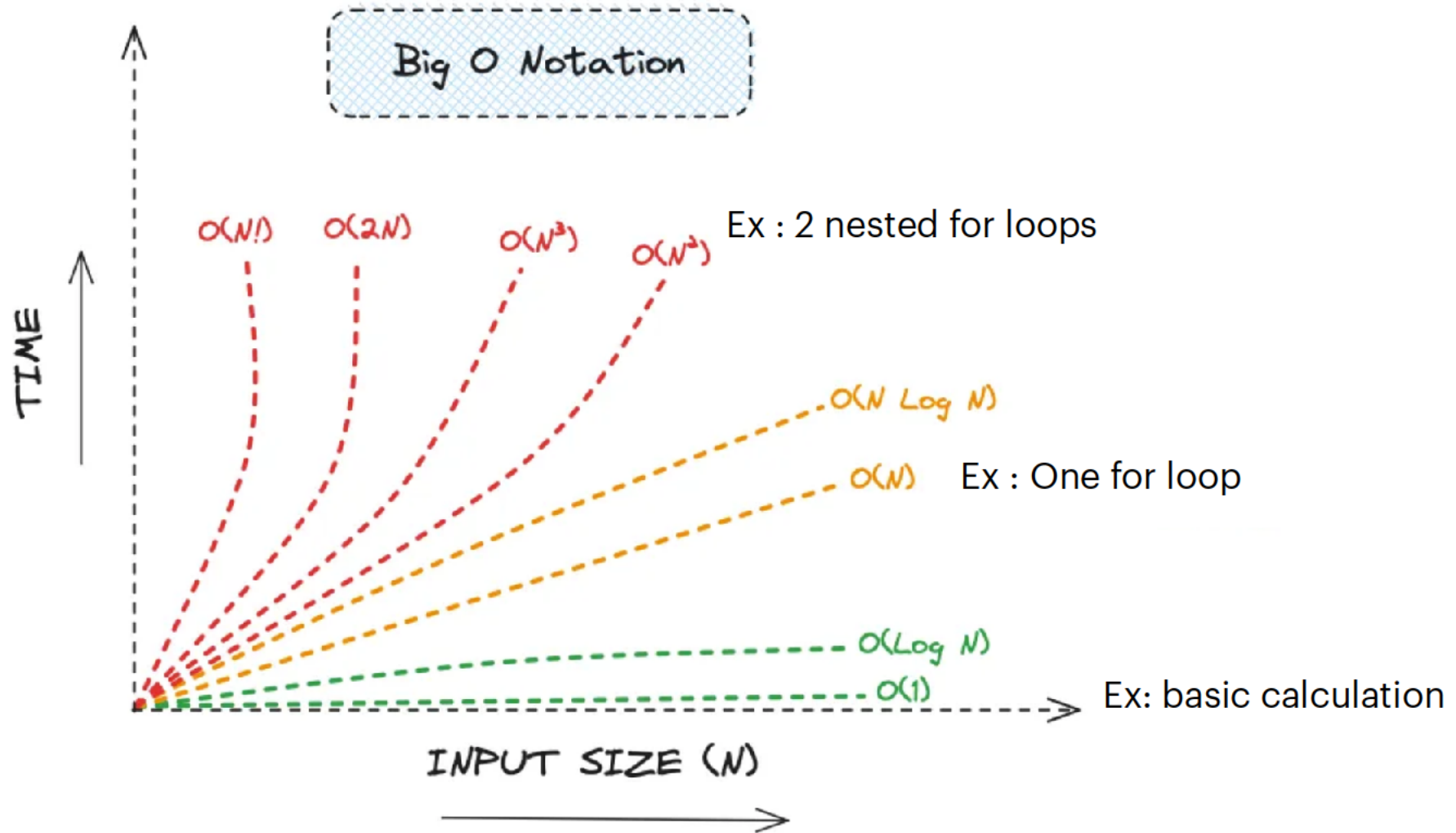
Asymptotic Notation

O-Notation



Asymptotic Notation

O-Notation



Asymptotic Notation

Ω -Notation

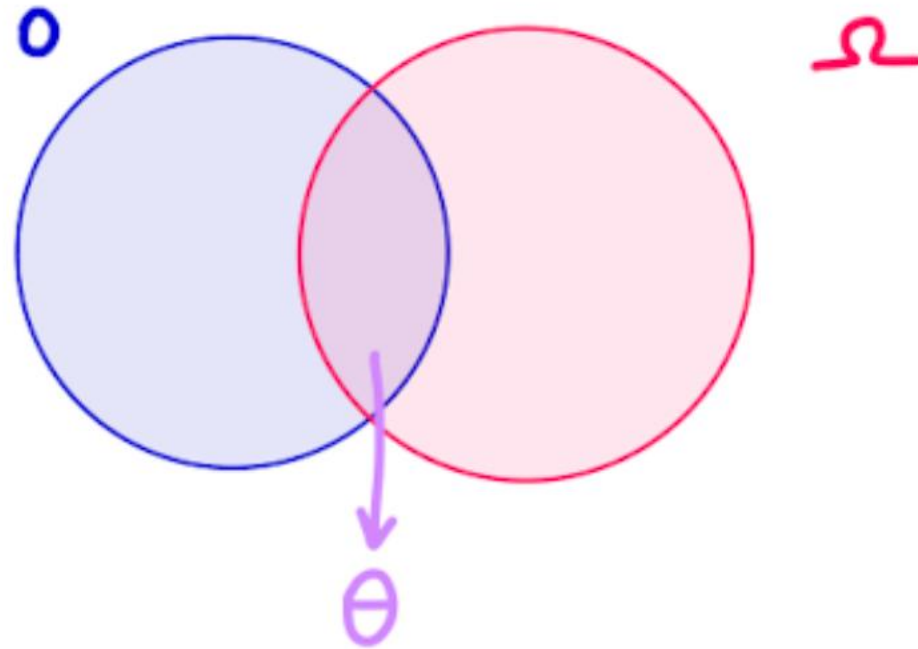
- Describes the lower-bound of an algorithm
- To express the best-case running time
- “ O-Notation reversed ”

$$\Omega: \quad \sqrt{n} \leq O(n) \\ n \geq \Omega(\sqrt{n})$$

- n represents the amount of data that we're passing in
- We don't care about the differences, when those differences are constant values. We care about n

Asymptotic Notation

θ -Notation



Asymptotic Notation

Definitions

Definition 1 (*O*-Notation). For $f : N \rightarrow \mathbb{R}^+$,

$$O(f) := \{g : N \rightarrow \mathbb{R}^+ \mid \exists C > 0 \forall n \in N \ g(n) \leq C \cdot f(n)\}.$$

$$\mathcal{O}(n^2) = \{n, \log(n), \sqrt{n}, \log^2(n), 1, 100n^2, \dots\}$$

Asymptotic Notation

Definitions

Definition 1 (O -Notation). For $f : N \rightarrow \mathbb{R}^+$,

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Definition 1 (Ω -Notation). For $f : N \rightarrow \mathbb{R}^+$,

$$\Omega(f) := \{g : N \rightarrow \mathbb{R}^+ \mid f \leq O(g)\}.$$

We write $g \geq \Omega(f)$ instead of $g \in \Omega(f)$.

Definition 2 (Θ -Notation). For $f : N \rightarrow \mathbb{R}^+$,

$$\Theta(f) := \{g : N \rightarrow \mathbb{R}^+ \mid g \leq O(f) \text{ and } f \leq O(g)\}.$$

We write $g = \Theta(f)$ instead of $g \in \Theta(f)$.

In other words, for two functions $f, g : N \rightarrow \mathbb{R}^+$ we have

$$g \geq \Omega(f) \Leftrightarrow f \leq O(g)$$

and

$$g = \Theta(f) \Leftrightarrow g \leq O(f) \text{ and } f \leq O(g).$$

Asymptotic Notation

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Theorem 1 (Theorem 1.1 from the script). Let N be an infinite subset of $\mathbb{N}$ and $f : N \rightarrow \mathbb{R}^+$ and $g : N \rightarrow \mathbb{R}^+$.

- If $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$, then $f \leq O(g)$, but $f \neq \Theta(g)$.
- If $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = C \in \mathbb{R}^+$, then $f = \Theta(g)$.
- If $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$, then $f \geq \Omega(g)$, but $f \neq \Theta(g)$.

Asymptotic Notation

Mini cheat-sheet

$$\lim_{n \rightarrow \infty} : 1 < \log(\log(n)) < \log(n) < \sqrt{n} < n < n \cdot \log(n) < n \cdot \sqrt{n} < \underbrace{n^2}_{n^x} < \underbrace{2^n}_{x^n} < n! < n^n$$

Sums

$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Geometric Series :

$$\sum_{k=0}^n q^k = \frac{q^{n+1} - 1}{q - 1} \quad \sum_{k=0}^3 3^k = 3^0 + 3^1 + 3^2 + 3^3 = \frac{3^4 - 1}{3 - 1} = 40$$

Factorial

$$\frac{n}{2}^{\frac{n}{2}} \leq n! \leq n^n$$

$$\Theta : n \cdot \ln n = \Theta(\ln(n!))$$

Asymptotic Notation

Mini cheat-sheet

This is a work in progress!

$$\lim_{n \rightarrow \infty} : 1 < \log(\log(n)) < \log(n) < \sqrt{n} < n < n \cdot \log(n) < n \cdot \sqrt{n} < \underbrace{n^2}_{\downarrow} < \underbrace{2^n}_{\downarrow} < n! < n^n$$

$$n^x < x^n \text{ (x being fixed)}$$

Sums

$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

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Factorial

$$\frac{n}{2}^{\frac{n}{2}} \leq n! \leq n^n$$

From Exercise Sheet 1 : $1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$.

Asymptotic Notation

Mini cheat-sheet

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Factorial

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Asymptotic Notation

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$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^n i^k = \Theta(n^{k+1})$$

Factorial

$$\frac{n}{2}^{\frac{n}{2}} \leq n! \leq n^n$$

Geometric Series :

$$\sum_{k=0}^n q^k = \frac{q^{n+1} - 1}{q - 1} \quad \sum_{k=0}^3 3^k = 3^0 + 3^1 + 3^2 + 3^3 = \frac{3^4 - 1}{3 - 1} = 40$$

Asymptotic Notation

log

log- Rules:

$$\begin{aligned}\log_a(bc) &= \log_a(b) + \log_a(c) \\ \log_a(b^c) &= c \log_a(b) \\ \log_a(1/b) &= -\log_a(b) \\ \log_a(1) &= 0 \\ \log_a(a) &= 1 \\ \log_a(a^r) &= r \\ \log_{1/a}(b) &= -\log_a(b) \\ \log_a(b) \log_b(c) &= \log_a(c) \\ \log_b(a) &= \frac{1}{\log_a(b)} \\ \log_{a^m}(a^n) &= \frac{n}{m}, \quad m \neq 0\end{aligned}$$

$$\log_b a = \frac{\log_d a}{\log_d b}$$

- From now on we use :
 - log for $\log_2(x)$
 - ln for $\log_e(x)$

► Converting exponentials from base b to base e

Q3: How do we convert b^x to $e^{(\text{something})}$?

A3: Using $b = e^{\ln b}$ we have the conversion formula:

$$b^x = (e^{\ln b})^x = e^{(\ln b) \cdot x}.$$

Example 3 Rewrite $\sqrt[3]{7}$ as an exponential with base e . $x^{100} = (e^{\ln(x)})^{100} = e^{\ln(x) \cdot 100}$

$$\sqrt[3]{7} = 7^{\frac{1}{3}} = (e^{\ln 7})^{\frac{1}{3}} = e^{\frac{1}{3} \ln 7}$$

$$\begin{aligned}\log(m^3) &= \log(m \cdot m \cdot m) \\ \log(m)^3 &= \log m \cdot \log m \cdot \log m = \log^3(m)\end{aligned}$$

work with
* log computation: $\frac{\log(m^3)}{\log(m)^3} = \frac{3 \log m}{\log(m)^3} = \frac{3}{\log(m)^2} \rightarrow 0$

Why log bases don't matter in asymptotic growth?

- The general idea is that different bases for the logarithm change the function only by a constant
 - $\log_2(n)$, $\log_{10}(n)$, $\ln(n) \in \Theta(\log n)$.
 - So they don't change the asymptotic behavior
 - If they're not used as an exponent (But there ^{3} and ^{4} are also different)
- Why exactly ?
 - Changing the base

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} = \frac{1}{\log_a(b)} \cdot \log_a(x)$$

Asymptotic Notation

Exam question

FS20

/ 4 P

a) *Landau notation:* Fill out the quiz about asymptotic notation below. You get 1P for a correct answer, -1P for a wrong answer, 0P for a missing answer. You get at least 0 points in total.

claim	true	false
$(2n + n^2 + 3)^2 = \Theta(n^4)$	☐	☐
$\frac{n}{\log n} \leq \mathcal{O}(\sqrt{n})$	☐	☐
$\log(n!) = \Theta(n \log n)$	☐	☐
$\sum_{i=1}^{\log_5 n} 5^i \geq \Omega(n \log n)$	☐	☐

Asymptotic Notation

Exam Tipps

- Try to gain an intuition
- Mind the time! (Spend 1 min on each exercise at max)
- Don't forget the subresults of the exercise sheets
- Our "cheat sheet" to the rescue!
- Remember the definitions

Non-bonus exercises of sheet 1

Exercise 1.2 *Sum of reciprocals of roots.*

Consider the following claim:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq \sqrt{n}.$$

A student provides the following induction proof. Is it correct? If not, explain where the mistake is.

Base case: $n = 1$,

$$\frac{1}{\sqrt{1}} \leq 1, \text{ which is true.}$$

Induction hypothesis: Assume the claim holds for $n = k$, i.e.

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \sqrt{k}.$$

Induction step: Then, starting from the claim we need to prove for $n = k + 1$ and using logical equivalences:

$$\begin{aligned} \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq \sqrt{k+1} &\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \sqrt{k+1} - \frac{1}{\sqrt{k+1}} \\ &\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k+1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+1}} \\ &\iff \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}} \leq \frac{k}{\sqrt{k}} \leq \sqrt{k}, \end{aligned}$$

which is true, therefore the claim holds by the principle of mathematical induction.