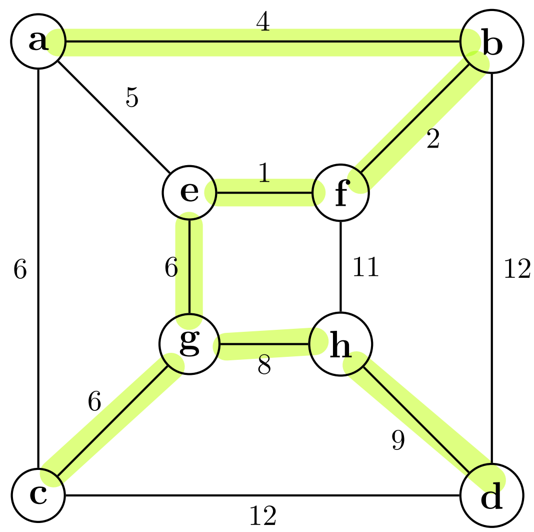
T1

/ 2 P

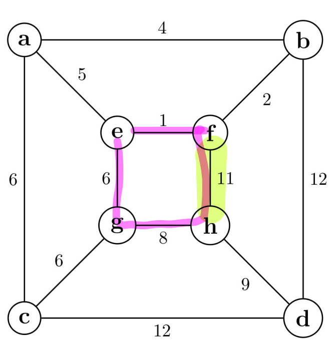
d) *Minimum Spanning Tree:* Consider the following graph:



i) Find a minimum spanning tree. You can either highlight its edges in the picture above, or draw the minimum spanning tree below. (If there is more than one minimum spanning tree, it suffices to just find one of them.)

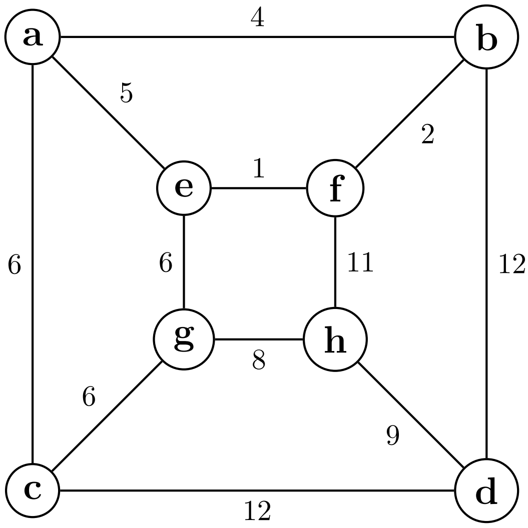
ii) Is there a minimum spanning tree containing the edge {f,h}? If yes, draw the corresponding minimum spanning tree. If no, give an argument.

No



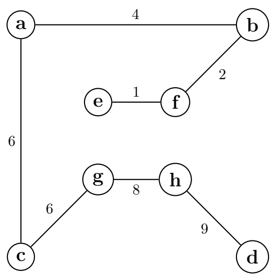
/ **2 P**

d) *Minimum Spanning Tree:* Consider the following graph:

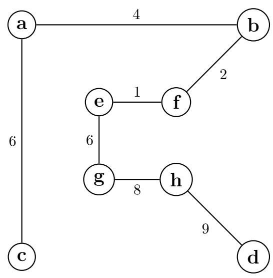


i) Find a minimum spanning tree. You can either highlight its edges in the picture above, or draw the minimum spanning tree below. (If there is more than one minimum spanning tree, it suffices to just find one of them.)

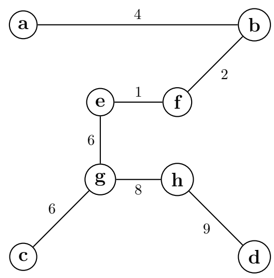
Solution:



or



or



ii) Is there a minimum spanning tree containing the edge {f,h}? If yes, draw the corresponding minimum spanning tree. If no, give an argument.

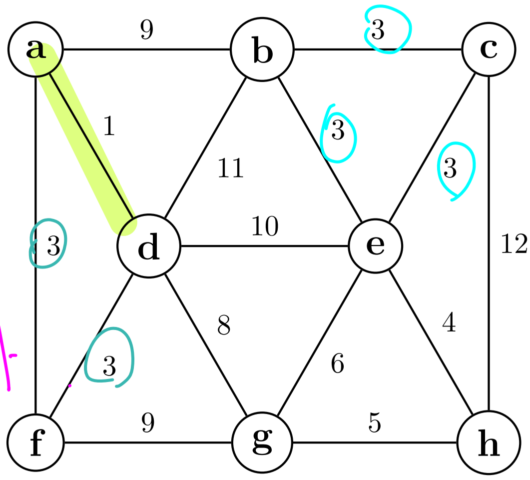
Solution: There is no MST containing the edge {f,h} since it is the heaviest edge of the cycle e,g,h,f and thus it cannot be part of any MST.

T2

/ 2 P d) *Minimum Spanning Tree:* Consider the following graph:

3

9



$\binom{3}{2}$

$\binom{2}{1}$

$\binom{3}{2} \cdot \binom{2}{1} = 6$

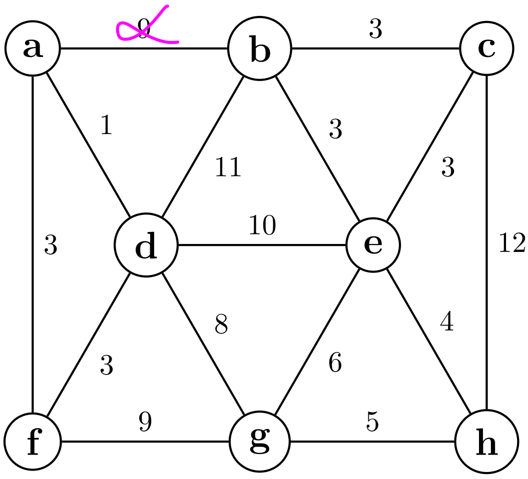
i) Find a minimum spanning tree. You can either highlight its edges in the picture above, or draw the minimum spanning tree below. (If there is more than one minimum spanning tree, it suffices to just find one of them.)

ii) How many different minimum spanning trees does the graph above have? You don't have to justify your answer.

/ 2 P

d) *Minimum Spanning Tree:* Consider the following graph:

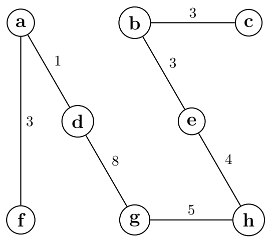
$\binom{3}{2}$



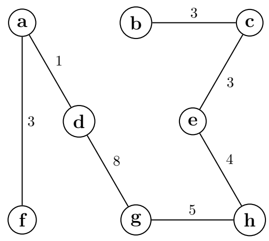
$\binom{2}{1}$

i) Find a minimum spanning tree. You can either highlight its edges in the picture above, or draw the minimum spanning tree below. (If there is more than one minimum spanning tree, it suffices to just find one of them.)

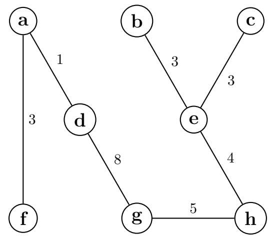
Solution:



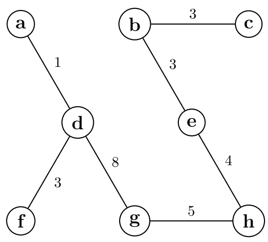
or



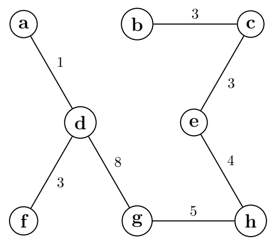
or



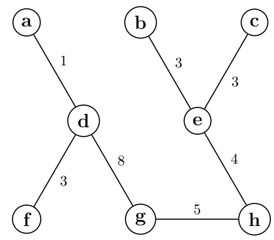
or



or



or



In order to find (one of) these MSTs, we can for example run Kruskal's algorithm and add the following edges: $\{a, d\}$; two of $\{b, c\}$, $\{b, e\}$ and $\{c, e\}$; one of $\{a, f\}$ and $\{d, f\}$; $\{e, h\}$; $\{g, h\}$; $\{d, g\}$.

ii) How many different minimum spanning trees does the graph above have? You don't have to justify your answer.

Solution:

6

To find this answer, we only need to consider weights that appear multiple times (every other edge is either part of every or of no MST). Out of the edges of weight 9, none of them is part of any MST since they are both the heaviest edge of a cycle ($a-b-e-g-d-a$ respectively $d-g-f-d$). For the edges of weight 3, we need to add exactly three of them to get an MST (we can add three while keeping it a tree and if we add any four we get a cycle since $\{a, d\}$ is part of every MST). More precisely, we need one of $\{a, f\}$ and $\{d, f\}$ and two of $\{b, c\}$, $\{b, e\}$ and $\{c, e\}$. This gives $\binom{2}{1} \cdot \binom{3}{2} = 2 \cdot 3 = 6$ possibilities.

P

/ 5 P b) *Many options?:* Let $G = (V, E)$ be an undirected, connected graph with $n = |V|$ vertices.

For each of the following statements, prove that they are correct, or give a counterexample and a brief argument why it is indeed a counterexample:

i) Let $k \in \mathbb{N}$ and suppose that $T_1, T_2, \ldots, T_k \subseteq E$ are (pairwise) disjoint spanning trees of G. Then $k \leq O(n)$.
(two spanning trees $T, T' \subseteq E$ are disjoint if $T \cap T' = \emptyset$, i.e., they share no edge.)

Correct.

Each tree has $n-1$ edges.

$\rightarrow$ $T_1 \cup T_2 \cup T_3 \cup \ldots \cup T_k$ has $\underbrace{k \cdot (n-1)}$

G has $\underbrace{O(n^2)}$ edges

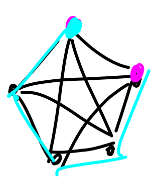
$k \cdot (n-1) \leq O(n^2)$

meaning $k \leq O(n)$ $\square$

ii) Let $v, w \in V$. The number of distinct paths $P \subseteq E$ from v to w in G is at most $O(n^{10})$.
(two subsets $P_1, P_2 \subseteq E$ are distinct if $P_1 \neq P_2$.)

False

K_n



$3 \cdot 2 \cdot 1$

$(n-2)! \geq n^{10}$

We have at least $\Omega((n-2)!)$ distinct paths $\geq n^{10}$

/ 5 P

b) *Many options?:* Let $G = (V, E)$ be an undirected, connected graph with $n = |V|$ vertices.

For each of the following statements, prove that they are correct, or give a counterexample and a brief argument why it is indeed a counterexample:

i) Let $k \in \mathbb{N}$ and suppose that $T_1, T_2, \ldots, T_k \subseteq E$ are (pairwise) disjoint spanning trees of G. Then $k \leq O(n)$.
(two spanning trees $T, T' \subseteq E$ are disjoint if $T \cap T' = \emptyset$, i.e., they share no edge.)

Solution:

This is correct. Suppose $T_1, T_2, \ldots T_k \subseteq E$ are pairwise disjoint spanning trees of G. Each tree has exactly $n-1$ edges. Therefore, using disjointness, the union $T_1 \cup T_2 \cup \ldots \cup T_k$ of all trees contains $k(n-1)$ edges. We know that G contains $O(n^2)$ edges in total, and we conclude that $k(n-1) \leq O(n^2)$, meaning $k \leq O(n)$.

ii) Let $v, w \in V$. The number of distinct paths $P \subseteq E$ from v to w in G is at most $O(n^{10})$.
(two subsets $P_1, P_2 \subseteq E$ are distinct if $P_1 \neq P_2$.)

Solution:

This is incorrect. For a counterexample, consider the complete graph $G = K_n$ with $n \geq 5$ vertices (i.e., the graph which contains all possible edges). Let v, w be any two vertices of G. Then, any sequence $v, u_1, u_2, u_3, \ldots u_{n-2}, w$, where the u_i are distinct vertices of G not equal to v, w gives rise to a valid path from v to w in G. Furthermore, these paths are all distinct. Thus we have at least $\Omega((n-2)!) \geq \Omega(2^n)$ distinct paths, which is more than $O(n^{10})$.